

W. Simpson
ELEMENTS
OF
GEOMETRY;
WITH THEIR
Application to the Mensuration
of SUPERFICIES and SOLIDS,
TO THE
Determination of the MAXIMA and MINIMA
of Geometrical Quantities,
AND TO THE
Construction of a great Variety of GEOME-
TRICAL PROBLEMS.

K
By *THOMAS SIMPSON*, F. R. S.
And Member of the Royal Academy of Sciences at
STOCKHOLM.

The FIFTH EDITION,
Carefully Revised.

L O N D O N:

Printed by *Luke Hansard*, Great Turnstile, Lincoln's-Inn Fields,
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in the Strand.

1800.



TO THE HONOURABLE

Charles Frederick, Esq;

Surveyor-General of His MAJESTY'S
ORDNANCE, &c. &c. &c.

HONOURABLE SIR,

THE subject of the sheets which I here beg leave to lay before you, is of so much consequence to mankind, as justly to claim the regard and sanction of the Great. Geometry is, not only a most accurate, but a very extensive science, whose application and great utility, as well in the arts of peace as of war, are well known to You.

But though this work, if the manner in which it is executed be correspondent to the importance of the subject, may not want sufficient merit to render it worthy of the approbation of a Gentleman, who, amidst a multiplicity of public employments, preserves an undiminish'd ardor for the sciences,

iv DEDICATION.

and a knowledge of the works of art and nature; yet I have, Sir, still farther motives for this address: Your great influence and zeal to promote the good of an institution under which I am placed; and the favours that I have received at your hands, make me earnest to embrace this opportunity of testifying publickly, that I am,

HONOURABLE SIR,

With great Respect,

Your much obliged,

and most obedient

humble Servant,

Thomas Simpson.

Royal Academy,
March 3, 1760.

21 MA 66

P R E F A C E.

*M*Y design in writing upon the subject of Geometry, was to open an easy way for young beginners to arrive at a proficiency in that useful science; without either being obliged to go thro' a number of unnecessary propositions, or having recourse to the ungeometrical methods of demonstration, that abound in most modern compositions of this nature.

The difficulty of the undertaking, I was not unapprised of; and objections occurred that were not easy to be removed: Nevertheless, I have grounds to hope, from the reception my first attempt has met with, that my endeavours have not been entirely unsuccessful. No pains have, indeed, been spared to render the work useful: And I flatter myself, that the spirit and rigour of demonstration, so essential to the subject, are also tolerably well preserved; though I have not been so intent to guard against the attack of Criticks, as to lose sight of my main design of furnishing a plain, easy institution for learners: Yet I have strong hopes, that there will not be found in these sheets, any inaccuracies, or oversights, that are absolutely unpardonable. To expect a faultless piece is impossible: And I well know that the most elaborate and best-approved systems of Geometry extant, are not without many imperfections. But, were the smallest imperfection to be a real fault, my ambition would rather be, to shew some degree of judgment, by avoiding a multitude of such faults, than by exposing and magnifying the flaws of other writers. It is more easy to see a fault, than to avoid one: And those men who are the most sanguine to distinguish themselves at the expence of others, are

generally observed to stand in need of greater indulgencies, than even the persons whom they unmercifully attack. But I shall put an end to this digression by pointing out one objection, that may be brought against this work; which is, that in demonstrations admitting of several cases, the most easy ones are sometimes omitted; and that the converse of some propositions is not at all demonstrated. But this, I conceive, will be found a real advantage to the learner; without which, it would have been impossible to have comprised the Elements in the compass they now take up. Besides, the greatest part of the demonstrations omitted being such as may be inferred from those given, by means of Axioms only; they may, therefore, be easily supplied by any reader, should they happen to become necessary, which I have scarce ever found to be the case. But, even allowing this to be a defect, it is abundantly compensated by the extensive application given in the three last sections; which is infinitely more useful, in itself, and more necessary to the forming an able Geometrician, than any thing of the kind we have been speaking of.

In this, second, edition (which is, in a manner, a new work) many considerable alterations and additions have been made. The order of some of the first propositions is changed: And some difficult propositions in the second book are rendered more plain. In the fourth book several new Theorems on proportions are added. The solid Geometry is now connected with the plane, and is demonstrated with the same accuracy. The mensuration of Superficies and Solids is also more explicitly handled; and the demonstration of the several rules is here established on a better foundation, than even in authors who have wrote professedly on the subject. The Maxima and Minima, and the construction of Geometrical Problems, are likewise considerably extended and improved. And, at the end,

Notes geometrical and critical, very useful to improve the judgment of young students, are now added.

But, whilst I am talking of improvements and matters of criticism, I am called upon to answer to a charge, which, should it appear to deserve credit, would indeed leave me but little room to pass myself upon the world for a judge in these matters. As the gentleman by whom I stand accused, is known to the world by his holding one of the most considerable mathematical posts in the kingdom; I shall, in order to do all due honour to the manner and importance of his writing, give you his own words.

“ There has lately been published a book under the title of *Elements of Plane Geometry*, designed for the use of schools, which is an incorrect copy of the first eight sections of this work, lent the pretended author on a particular occasion, and printed in a spurious manner, without my knowledge or consent; an action too scandalous for any man of honour to be guilty of. The Editor imagined, I suppose, that the changing some propositions, and mangling the demonstrations of others, was a sufficient disguise to make it pass for his own performance; but how far this will justify such a piece of piracy, must be left to the judgment of the publick.”

Were I to attempt to describe the ideas excited in my mind by the singular modesty of this important and solemn appeal to the publick, I should be at a loss for fit words to express them, without transgressing the bounds of decency. But I hope that I have not deserved so ill of the publick, to be thought capable of acting so very bumble a part, as that of copying from this author, and of mangling his demonstrations, in order to make them pass for my own.—That a manuscript of his (containing between 20 and 30 of the principal

principal Theorems in Geometry, extremely ill digested came into my hands, is indeed true; but it was not lent me, but forced upon me, by himself (the very first night after my removal to Woolwich) in virtue of an article in the original rules and instructions for the Academy; whereby it is ordered, that the second master shall teach Geometry under the direction of the first master. But this well intended article, which has been made subservient to the purposes of ignorant tyranny, and daring calumny, has since, in consequence of a publick examination, been annulled by an express order of the Master-General of the Ordnance.—I could mention some particulars, supported by good authority, that occurred in the course of that examination, which would but ill agree with the importance he assumes in his confident accusation; but I do not think it worth while: This Gentleman has, himself, by his different publications, so well convinced the world of his abilities, as to render any farther comment on that head intirely unnecessary and ineffectual.

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ADVER-

ADVERTISEMENT.

AS in every work of this nature, designed to contain whatever may be most requisite to the forming of a regular and complete system of Geometry, a number of propositions must necessarily have a place, whose chief use and application lie in the higher branches of the Mathematics; and there being many persons, particularly young gentlemen in publick schools, who want to learn so much Geometry only, as is necessary to give them a proper introduction into the practical and most common applications thereof; such as Mensuration, Trigonometry, Navigation, Fortification, Perspective, &c. For these reasons, I thought that it might be of service, to point out to such Readers, what propositions in these elements may be omitted, as least useful to them; without either hurting the connection, or taking away from the evidence of the other demonstrations. The numbers of these propositions, in the several books, are as follow.

In Book I. the 6, 17, 19, 21, 22, 23, and 29th.

In Book II. the 4, 5, 10, 11, 12, 13th, and the 2d Corol. to the 9th.

In Book III. the 4, 5, 6, 7, 8, 9, 15, 18, 19, 20, 25, 26, 27, and 28th.

In Book IV. the 4, 5, 6, 9, 11, 13, 16, 17, 20, 21, 22, 23, 25, 26, 27, 28, and 29th.

In

In Book V. the 1, 2, 16, 17, 18, 19, 20, 25, 26, 28, and 31st.

In Book VI. the two or three first propositions only, need be read; except by those who are concerned in surveying and dividing of lands; to whom the whole Book will be highly useful,

Also, with regard to the seventh book, if *Perspective* be the only application in view (which I have known frequently to be the case) the 1st, 2d, 4th, and 12th propositions may suffice. But if a more general idea of the properties of intersecting planes should be required, such as is necessary in the doctrine of solids and spheric geometry; then all the propositions, to the 12th, ought to be taken.

The 17th, 19th, 20th, 21st, 22d, and 23d propositions of this seventh book should also be read by those who would be able to find the content and proportion of solid bodies; as should, likewise, the whole eighth book; except, perhaps, the first and ninth propositions, together with the three first lemmas; which may be thought too plain, by those who are not very solicitous about geometrical rigour, to need a demonstration.

An INDEX or TABLE referring to the places in these Elements, where all the most material propositions in the first six, and in the eleventh and twelfth books of Euclid, are demonstrated.

<i>Euclid.</i>	<i>These</i> <i>El.</i>	<i>Euclid.</i>	<i>These</i> <i>El.</i>	<i>Euclid.</i>	<i>These</i> <i>El.</i>	<i>Euclid.</i>	<i>These</i> <i>El.</i>
<i>Book I.</i>	<i>P. B.</i>	<i>B. II.</i>	<i>P. B.</i>	<i>B. V.</i>	<i>P. B.</i>	<i>B. XI.</i>	<i>P. B.</i>
<i>Prop. 5</i>	12.1	<i>P. 11</i>	19.5	<i>P. 16</i>	2.4	<i>Pr. 5</i>	<i>Cor. to</i>
6	18.1	12.7		17			2.7
8	14.1	13.3	10.2	18	3.4	6	4.7
9	5.5	14	9.6	19		8	5.7
10	6.5	<i>B. III.</i>	<i>P. B.</i>	22	5.4	9	8.7
11	3.5			24	6.4	10	9.7
12	4.5	<i>Pr. 3</i>	2.3	25	3.4	12	3.7
13	1.1	7.7		<i>B. VI.</i>	<i>P. B.</i>	14	7.7
15	3.1	8	5.3	<i>Pr. 1</i>	7.4	15	10.7
18		11	8.3	2	12.4	16	11.7
19	13.1	12	7.3	3	18.4	18	6.7
21	23.1	14	3.3	4	14.4	19	13.7
22	8.5	15	4.3	5	17.4	24	16.7
23	7.5	17	21.5	6	15.4	25	21.7
24	21.1	20	10.3	7	16.4	28	
26	15.1	21	11.3	8	19.4	29	
27		22	17.3	9	11.5	30	
28	8.1	25	18.3	10	15.5	31	17, 10 and 20 of the 7th, and Cor 3, 8.8
29	7.1	31	16.3	11	12.5	32	21.7
31	9.5	32	14.3	12	13.5	33	9.8
32	9.1	33	22.5	13	14.5	34	23.7
33	26.1	34	25.5	14	<i>Cor. to</i>	35	14.7
34	24.1	35	21.3	15	25.4	37	14.7
35	2.2	36	22.3	16			25.7
36		37		17	10.4	<i>B. XII.</i>	<i>P. B.</i>
37	<i>Cor. to</i>	<i>B. IV.</i>	<i>P. B.</i>	18	11.6	<i>Pr. 2</i>	3.8
38	2.2	<i>Pr. 2</i>	25.5	19	24.4	5	<i>Cor. 4</i>
41		3	26.5	20	26.4	6	<i>to 8.8</i>
42	6.6	4	24.5	22	28.4	7	7.8
43	3.2	5	23.5	23	25.4	8	9.8
44	6.6	10.7		25	13.6	9	
45	7.6	11	28.5	27		10	
46	10.5	12	30.5	28	17.5	11	
47	8.2	15	29.5	29	18.5	12	
<i>B. II.</i>	<i>P. B.</i>	<i>B. V.</i>	<i>P. B.</i>	30	19.5	13	
<i>Pr. 1.</i>	5.2	4	4.4	31	29.4	14	
4	6.2	12	6.4	<i>B. XI.</i>	<i>P. B.</i>	15	
5	7.2	15	1.4	<i>Pr. 3</i>	1.7	18	<i>Cor. to</i>
				4	2.7		11.8

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ELEMENTS
OF
GEOMETRY.

BOOK I.

DEFINITIONS.

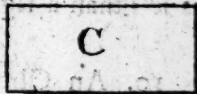
1. **G**EOMETRY is that science, by which we compare such quantities together as have extension.

Extension is distinguished into length, breadth, and thickness.

2. A Line is that, which has length without breadth.

The terms, bounds, or extremes of a Line are points.

3. A Surface is that, which has length and breadth, only, as
C.



The bounds of a Surface are lines.

B

4. A

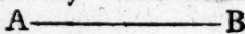
Elements of Geometry.

4. A Solid is that, which has length, breadth, and thickness, as D.



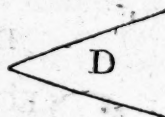
The bounds of a Solid are surfaces.

5. A Right (or strait) line is that, which lies evenly between its extremes, or which every-where tends the same way, as AB.

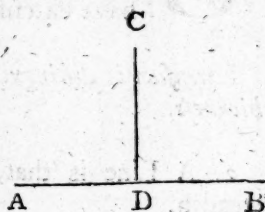


6. A Plane-surface is that, which is every-where perfectly flat and even, or which touches, in every part, any right-line extended between points any-where taken in that surface.

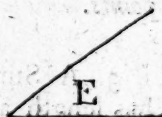
7. An Angle is the inclination, or opening of two right-lines meeting in a point, as D.



8. When one right-line DC, standing upon another AB, makes the angles on both sides equal, those angles are called right-angles; and that line CD is said to be perpendicular to the other AB on which it infits.



9. An Acute-angle is that, which is less than a right-angle, as E.



10. An Obtuse-angle is that, which is greater than a right-angle, as F

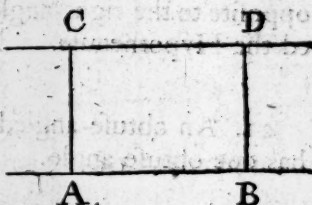


11. The

11. The distance of two points, is the Right-line reaching from the one to the other.

12. The distance of a point from a line, is a Right-line drawn from that point, perpendicular to, and terminating in, the line given.

13. Parallel (or equidistant) right-lines AB, CD are such, which being in the same plane-surface, if infinitely produced, would never meet.



14. A Figure is a bounded space, and is either a surface, or a solid.

15. A right-lined plane Figure is that, formed in a plane-surface, whose terms, or bounds, are right-lines.

16. All plane Figures bounded by three right-lines, are called Triangles.

17. An equilateral Triangle is that, whose bounds or sides are all equal, as A.



18. An isosceles Triangle is, when two sides are equal, as B.

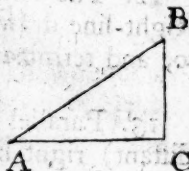


Elements of Geometry.

19. A scalene Triangle is, when all the three sides are unequal, as C.



20. A right-angled Triangle is that, which has one right-angle, as ACB; whereof the side AB opposite to the right-angle, is called the Hypothenuse.

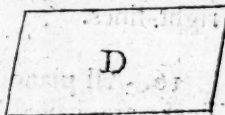


21. An obtuse-angled Triangle is that, which has one obtuse angle.

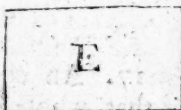
22. An acute-angled Triangle is that, which has all its angles acute.

23. Every plane Figure bounded by four right-lines, is called a Quadrangle, or Quadrilateral.

24. Any Quadrangle, whose opposite sides are parallel, is called a Parallelogram, as D.



25. A Parallelogram, whose angles are all right-ones, is called a Rectangle, as E.



26. A Square is a parallelogram whose sides are all equal, and its angles all right-ones, as F.



27. A Rhombus is a parallelogram whose sides are all equal, but its angles not right, as G.



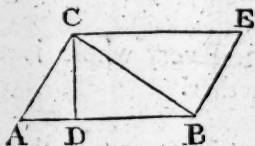
28. All

Book the First.

28. All other four-sided figures, besides these, are called Trapeziums.

29. A right-line joining any two opposite angles of a four-sided figure, is called a Diagonal.

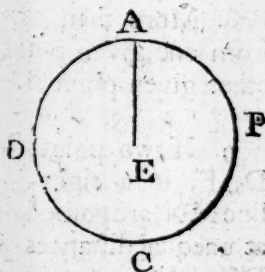
30. That side AB upon which any parallelogram ACEB, or triangle ACB is supposed to stand, is called the base; and the perpendicular CD falling thereon from the opposite angle C, is called the altitude of the parallelogram, or triangle.



31. All plane figures contained under more than four sides, are called Polygons; whereof those having five sides, are called Pentagons; those having six sides, Hexagons; and so on.

32. A Regular Polygon is one whose angles, as well as sides, are all equal.

33. A Circle is a plane figure, bounded by one curve-line APCD, called its circumference, everywhere equally distant from a point E within the circle, called the center thereof.



34. The Radius of a circle, is the distance of the center from the circumference, or a right-line EA drawn from the center to the circumference.

Elements of Geometry.

A X I O M S, or Self-evident Truths.

1. Things, equal to one and the same thing, are also equal to each other.

2. Every whole is greater than its part.

3. Every whole is equal to all its parts taken together.

4. If to equal things, equal things be added, the wholes will be equal.

5. If from equal things, equal things be taken away, the remainders will be equal.

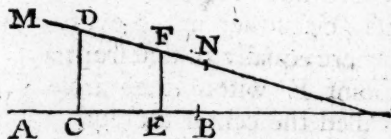
6. If to, or from unequal things, equal things be added, or taken away, the sums, or remainders, will have the same difference, as the unequal things first proposed.

7. All right-angles are equal to one another.

8. More than one right-line cannot be drawn from one given point A to another given point B.

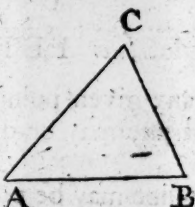
A ————— B

9. If two points D, F, in a right-line MN, are posited at unequal distances DC, FE, from another right-line AB in the same plane-surface; those two lines, being infinitely produced, on the side of the least distance EF, will meet each other.



10. If

10. If two right-lines CA, CB, making an angle C, be respectively equal to two other right-lines FD,



FE, making an angle F, and the angles which they make C, and F, be likewise equal; the right-lines AB, DE joining their extremes will be equal, and the two triangles ACB, DFE equal in all respects.

If this should not appear sufficiently evident for an axiom; conceive the triangle DFE to be removed, and so applied to the triangle ABC, that the point F may coincide with C, and the side FD fall upon the side CA; then, because FD is supposed equal to CA, the point D will also fall upon A. And, the angle F being equal to the angle C, the side FE will fall upon CB; and consequently the point E upon the point B, because FE is supposed equal to CB. Therefore, seeing all the bounds of the two triangles coincide, it is manifest, that not only the bases AB, DE, but the angles opposite to the equal sides, are also equal.

When all the four lines CA, CB, FD, FE are equal; the triangle DFE, being *contrariwise* applied to ACD so that FE may coincide with CA, will, *also*, agree with the triangle ACB (as is manifest from the reasoning above): and so, the angle E (as D did before) now coinciding with the angle A, the two angles E and D must necessarily be equal to each other, in this case, where the triangle DFE is an isosceles one.

Elements of Geometry.

POSTULATES, or PETITIONS.

1. That, from any given point, to any other given point, a right-line may be drawn.

2. That, a right-line may be produced, or continued out, at pleasure.

3. That, from any point as a center, with a radius equal to any right-line assigned, a circle may be described.

4. That, a right-line may be drawn perpendicular to another, at any point assigned; and that it is also possible for to make a right-line, or a right-lined angle, equal to any right-line, or right-lined angle assigned, or to the half thereof.

This fourth Postulate is added, more for the sake of making the proper references, than through absolute necessity; since, what is here barely assumed as possible, is effected, and actually demonstrated, in the beginning of the Fifth Book, intirely independent of every thing but Axioms and the other Postulates, above laid down. It may also be proper to note here, that, though these Postulates are not always quoted, it will be easy to perceive where, and in what sense, they are to be understood.

NOTES and OBSERVATIONS, with the significations of Signs used in this Tract.

A PROPOSITION is, when something is either proposed to be done, or to be demonstrated, and is either a Problem, or a Theorem.

A PROBLEM is, when something is proposed to be done.

A THEO-

A THEOREM is, when something is proposed to be demonstrated.

A LEMMA is, when some premise is demonstrated, in order to render the thing in hand the more easy.

A COROLLARY is, a consequent truth, gained from some preceding truth, or demonstration.

A SCHOLIUM is, when remarks and observations are made upon something going before.

The signification of SIGNS.

The sign $=$, denotes that the quantities betwixt which it stands, are equal.

The sign $>$, denotes that the quantity preceding it, is greater than that which comes after it.

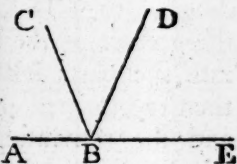
The sign $<$, denotes that the quantity preceding it, is less than that which comes after it.

The sign $+$, denotes that the quantity which it precedes, is to be added.

The sign $-$, denotes that the quantity which it precedes, is to be taken away or subtracted.

A figure, or number, prefixed to any quantity, shews how often that quantity is to be taken, or repeated; as 5 A shews, that the quantity represented by A, is to be taken 5 times.

When several angles are formed about the same point (as at B), each particular angle is described by three letters, whereof the middle one shews the angular point, and the other two, the lines that form the angle: thus CBD or DBC signifies the angle formed by the lines CB and DB.



When,

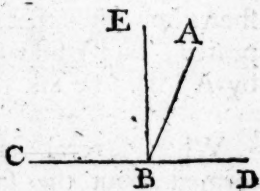
When, in any demonstration, you meet with several quantities joined the one to the other continually by the mark of equality ($=$), the conclusion drawn from thence, is always gathered from the first and last of them; which are equal to each other, by virtue of the first axiom. Thus if $A=B=C=D$, then will the first (A) and the last (D) be equal to each other.

Also, when in the quotations you meet with two numbers, the first shews the proposition, and the second the book. Moreover, Ax. denotes axiom; Post. postulatum; Def. definition; Hyp. hypothesis. Note also, that, whenever the word *Line* occurs, without the addition of either *right*, or *curved*, a right-line is always understood: and that, when a line is said to be drawn to, or from an angle, the angular point is meant.

THEOREM I.

A line (AB) standing upon another line (CD) makes with it two angles (ABC, ABD) which, taken together, are equal to two right-angles.

- If the angles ABC, ABD are equal, it is plain they make two right-angles^a; if unequal, let BE be perpendicular to CD^b, dividing the greater of them (ABC) into the parts EBC, EBA; then the former part EBC being a right-angle^a, and the remaining part EBA together with the whole lesser angle ABD, equal to another right-angle EBD^c; the whole, of both the proposed angles, taken together, must necessarily be equal to two right-angles^d.



COROL.

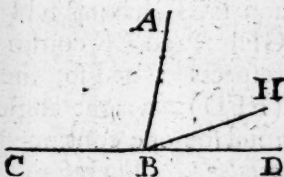
COROLLARY.

Hence all the angles at the same point (B) on the same side of a right-line (CD) are equal to two right-angles^a. ^a Ax. 3.

THEOREM II.

If one line (AB) meeting two others (BC, BD) in the same point (B), makes two angles with them (ABC, ABD) which together are equal to two right-angles; these lines (BC, BD) will form one continued right-line.

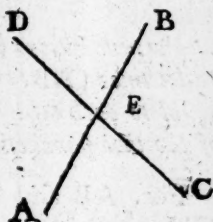
For, if possible, let BH, and not BD, be the continuation of the right line CB: then the angles ABC and ABH being = two right-angles^a = ABC and C B D^b. I. 1.
ABD^c; if from these equal quantities, ABC, common to both, be taken away, there will remain ABH = ABD^c; which is impossible^b. ^c Ax. 5.
^b Ax. 2.



THEOREM III.

The opposite angles (DEB, AEC), made by two lines (DC, BA) intersecting each other, are equal.

For $DEB + DEA =$ two right-angles^a = $AEC + DEA$; whence, by taking away DEA, common, there remains $DEB = AEC$ ^b.



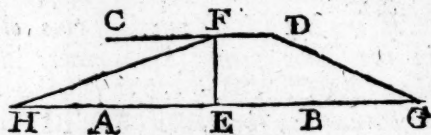
THEO.

THEOREM IV.

Two right-lines (AB, CD) perpendicular to one and the same right-line (EF), are parallel to each other.

If you say, they are not parallel; then let them, when produced out, meet in some point, as G.

In EA, produced (if necessary) let there be taken $EH =$



¹ Post. 4. EG^1 , and let

^m Post. 1. the right-line FH be drawn ^m. The triangles EHF and EGF, having $EH = EG$, the angle $HEF =$

ⁿ Def. 8. GEF^n , and EF common, are therefore equal in all

^o Ax. 10. respects ^o: and so, the angle EFH being $= EFG$

^p Hyp. $(EFD) =$ a right-angle ^p, $HFDG$ (as well as HEG)

^q z. 1. must be one continued right-line ^q: which is im-

^r Ax. 8. possible ^r. Therefore AB and CD. are parallels.

SCHOLIUM.

In this theorem, the *possibility* of parallel lines (or such, which being infinitely produced, in the same plane, can never meet) is demonstrated: for EF may be drawn perpendicular to AB^1 ; and CFD, again, perpendicular to EF^1 ; which last, it is demonstrated, will be parallel to AB.

THEOREM V.

Perpendiculars (EF, GH) to one (AB) of two parallel lines (AB, CD) terminated by those lines, are equal to each other; and also perpendicular to the other of the two parallels (CD).

For, AB and CD being parallel to each other,
^a Ax. 9. GH can neither be greater, nor less than EF^a ;
 and Def. and therefore must be equal to EF. If you say,
¹³. that EF is not perpendicular to CD; then let FM
¹ Post. 4. be perpendicular to EF^1 , meeting GH produced (if neces-

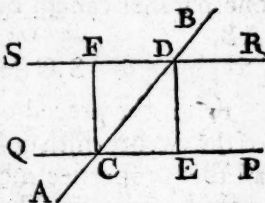
- For let the line HIG
 be perpendicular to CD :
 then, that line being also
 perpendicular to both AB
 and EF ^e, these last are
^d 4. 1. parallel to each other^d.



THEOREM VII.

A line (AB) intersecting two parallel lines (SR, QP) makes the alternate angles (SDC, PCD) equal to each other.

- Let CF and DE be perpendicular to QP , and SR ^e; ^e 5. 1. then these lines FC and DE
^f 4. 1. are likewise parallels^f; and
 so the triangles CFD and CDE , having the side CF
 $= DE$ ^e, $FD = CE$ ^e, and
^g Ax. 7. the angle $F = E$ ^g, they will also have the angle
^h Ax. 10. $FDC = ECD$ ^h.



COROLLARY I.

- Hence, a line intersecting two parallel lines,
 makes the angles (BDR , BCP) on the same side,
ⁱ 3. 1. equal to each other: for $BDR (=CDS')$ $= BCP$ ^k.
^k 7. 1.

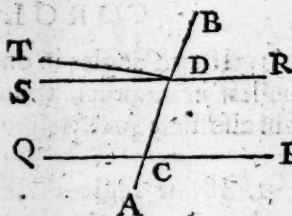
COROLLARY II.

- Hence, also, a line falling upon two parallel lines,
 makes the sum of the two internal angles ($SDC + QCD$) on the same side of it, equal to two right-angles: for the angle SDC being $= PCD$, and
^l 1. 1. $PCD + QCD =$ two right-angles^l; thence is
^m Ax. 4. $SDC + QCD =$ also to two right-angles^m.

THEOREM VIII.

If a line (AB) intersecting two other lines (PQ, RS), makes the alternate angles (DCP, CDS) equal to each other; then are those two lines parallel.

For, if possible, let
some other line DT, and
not DS, be parallel to
PQⁿ; then must CDT
= DCP^o = CDS^p;
which is impossible^q.



ⁿ Sch. to
4. 1.
^o 7. 1.
^p Hyp.
^q Ax. 2.

COROLLARY.

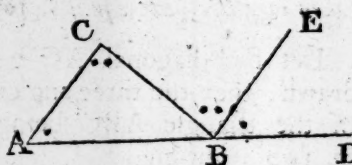
Hence, if a line falling on two others, makes
the angles (BDR, BCP) above them, on the same
side, equal to each other; then those two lines
are parallels: because SDC = BDR^r.

^r 3. 1.

THEOREM IX.

If one side (AB) of a triangle (ABC) be produced,
the external angle (CBD) will be equal to both the
internal opposite angles (A, C) taken together.

For, let BE be pa-
rallel to AC^s; then
will the angle C =
CBE^t, and the angle
A = DBE^u; there-
fore C + A = CBE
+ DBE^x = CBD^y.



^s Sch. to
4. 1.
^t 7. 1.
^u Cor. to
7. 1.
^x Ax. 4.
^y Ax. 3.

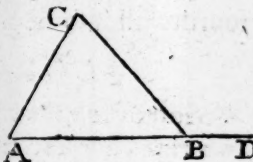
COROLLARY.

Hence the external angle of a triangle is greater
than either of the internal, opposite angles.

THEOREM X.

The three angles of any plane triangle (ABC) taken
together, are equal to two right-angles.

For, if AB be produced to
D, then C + A = CBD^z, to
which equal quantities let the
angle CBA be added, then
will C + A + CBA = CBD
+ CBA^a = two right-angles^b.



^z 9. 1.

^a Ax. 4.
^b 1. 1.

COROL-

COROLLARIES.

1. If two angles in one triangle, be equal to two angles in another triangle, the remaining angles
 * Ax. 5. will also be equal ^c.

2. If one angle in one triangle, be equal to one angle in another, the sums of the remaining angles will be equal ^c.

3. If one angle of a triangle be right, the other two taken together, will be equal to a right-angle.

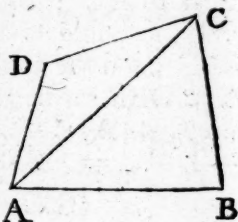
4. The two least angles, of every triangle, are acute.

THEOREM XI.

The four inward angles of a quadrangle (ABCD) taken together, are equal to four right-angles.

Let the diagonal AC be drawn; then the three angles of the triangle ABC being ^d = two right-angles ^d, and those of the triangle ACD equal also to two right-angles ^d; it follows that the sum of all the angles of both triangles, which make the four angles of the quadrangle, must be equal to four right-angles ^c.

* Ax. 4.



COROLLARY I.

Hence, if three of the angles be right ones, the fourth will also be a right-angle.

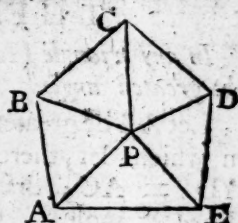
COROLLARY II.

Moreover, if two of the four angles, be equal to two right-angles, the remaining two together will likewise be equal to two right-angles.

SCHO-

SCHOLIUM.

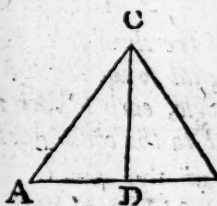
If from any point P, within a polygon ABCDE, lines be drawn to all the angles, so as to divide the whole into as many triangles APB, BPC, CPD, DPE, EPA, as the polygon has sides; the sum of all the angles of these triangles, (which together make up, or compose the angles of the polygon, over and above those about the point P) will be equal to twice as many right angles as the polygon has sides (by 10 1.) Therefore, seeing all the angles about the point P, whereby the angles of all the triangles exceed those of the polygon, are equal to four right angles, it is manifest, that all the angles of the polygon, taken together, will be equal to twice as many right angles, wanting four, as the polygon has sides.



THEOREM XII.

The angles (A, B,) at the base of an isosceles triangle (ABC) are equal to each other.

For, let the line CD bisect, or divide the angle ACB into two equal parts ACD, BCD, and meet AB in D: then the triangles ACD, BCD, having AC = BC, CD common, and the angle ACD = BCD, will also have the angle A = B.



Def. 18.

Hyp.

Ax. 10.

COROLLARY I.

Hence, the line which bisects the vertical angle of an isosceles triangle, bisects the base, and is also perpendicular to it.

C

COROL-

Elements of Geometry.

COROLLARY II.

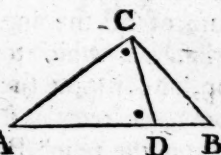
Hence it appears also, that every equilateral triangle is likewise equiangular.

THEOREM XIII.

In any triangle (ABC) the greatest side subtends the greatest angle.

Let AB be greater than AC;
in which let there be taken
 $AD = AC$; drawing CD.

The triangle ADC being
isosceles, the angles ACD and A



- ⁱ 12. 1. ADC are therefore equalⁱ; whence ACB, which exceeds the former of them, must also exceed the latter ADC^k, and consequently, much more exceed B, which is less than ADCⁱ.
^k Ax. 2.
ⁱ Cor. 1.
9. 1.

COROLLARY.

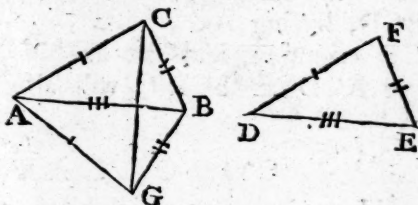
Hence, in any triangle, the side that subtends the greatest angle, is the greatest; because ACB cannot be greater than B, unless AB is greater than

^m 13. 1. AC^m.

THEOREM XIV.

If the three sides (AB, AC, CB) of one triangle, be equal to the three sides (DE, DF, FE) of another triangle, each to each respectively; then the angles opposed to the equal sides will also be equal.

Let the angle BAG = D,
 $AG = DF$, and
let GB and GC
be drawn; so
shall the triangles ABG and



- ^m Ax. 10. DEF be equal in all respects^m: therefore, AG
ⁿ Hyp. being = DF = ACⁿ, and BG = EF = BCⁿ,
the

*

the angle ACG is also $= AGC^o$; and BCG^o ^{12. 1.}
 $= BGC^o$; and consequently $ACB = AGB^o =$ ^{Ax. 4.}
 DFE : therefore the triangles ABC, DEF are equal ^{or 5.}
in all respects ^{m.}

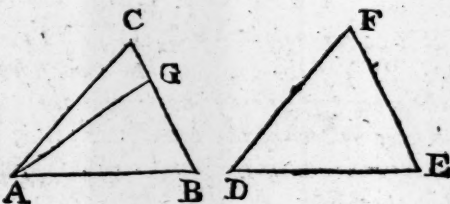
SCHOLIUM.

The demonstration of the last theorem, in obtuse-angled triangles, may admit of another case; which, however, is not necessary; because, if the triangle AGB (equal to DEF) be conceived to be formed on the longest side of ABC ; then, all the angles CAB, CBA, GAB, GBA being acute ^q, the line CG will, always, fall within the figure $ACBG$, ^{Cor. 4. to 10. 1.}
as in the present case. ^{r Ax. 2.}

THEOREM XV.

If two triangles (ABC, DEF) mutually equiangular, have two corresponding sides (AB, DE) equal to each other, the other corresponding sides will also be equal.

If you say BC is greater than EF ; from BC let a part BG be taken $= EF^s$, and let



AG be drawn. The triangles ABG, DEF having $AB = DE, BG = EF$, and $B = E$ (*by hypothesis*), will also have $BAG = D^s$; but $D = BAC^s$; ^{Ax. 10.}
therefore $BAG = BAC^s$; *which is impossible.* ^{Hyp. Ax. 1. and 2.}

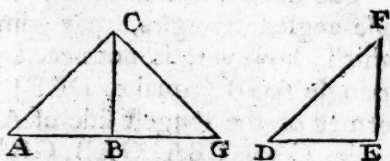
COROLLARY.

Hence, equiangular triangles, having any two corresponding sides equal, are equal to each other ^{x Ax. 10. 1.}

THEOREM XVI.

If two right-angled triangles (ABC, DEF) having equal hypotenuses (AC, DF), have two other sides (BC, EF) likewise equal; the remaining sides (AB, DE) will be equal, and the two triangles equal in all respects.

In AB produced, take $BG = ED$, and let GC be drawn: then, the triangles BCG and DEF, having BG

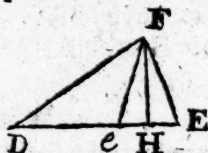
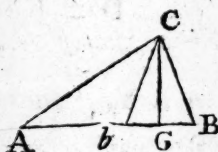


- ⁷ Hyp. $= ED$, $BC = EF$, and the angle $CBG = E$,
² Ax. 7. will also have the angle $G = D$, and $CG = DF$.
² Ax. 10. $= AC$: whence, the triangle ACG being isosceles,
¹² 1. the angle G , or D , will be $= A$; and consequently
^c Cor. 1. F also $= ACB$; therefore the triangles ABC and
¹⁰ 10.1. DEF, being mutually equiangular, and having
¹⁵ 1. $AC = DF$, they are equal in all respects.

THEOREM XVII.

If two triangles (ABC, DEF) having two sides (AC, BC) of the one equal to two sides (DF, EF) of the other respectively, have also the angles (A, D) subtended by two of the equal sides (BC, EF) equal to each other; and if the angles (B, E) subtended by the other equal sides, be either, both acute or both obtuse; then will the two triangles be equal in all respects.

Let CG and FH be perpendicular to AB and



- ^c Ax. 7. DE: then, the angle AGC being $= DHF$,
 $\S$ $A =$

$A = D$, and the side $AC = DF$, CG will also be f Hyp. $= FH$ g ; whence, CB being $= FE$ f , the angles s 15. 1. GBC and HEF are likewise equal h , and so, the h 16. 1. triangles ABC and DEF , being mutually equiangular i , and having the sides AC and DF equal, i Cor. 1. are equal in all respects k . to 10.

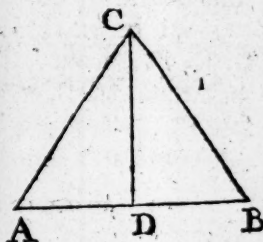
The demonstration is the same, when both the angles are obtuse, as in the triangles AbC , DeF : for, if $Cb (= CB = FE) = Fe$, the angles GbC and HeF being equal (*as before*) the angles AbC and DeF will likewise be equal k .

k 1. 1. and
Ax. 5. 1.

THEOREM XVIII.

If two angles (A, B) of a triangle (ABC) be equal, the sides (BC, AC) subtending them will likewise be equal.

Let CD bisect the angle ACB , and meet AB in D : then the triangles ACD , BCD being equiangular k , and having CD common to both, they will also have $AC = BC$ l .



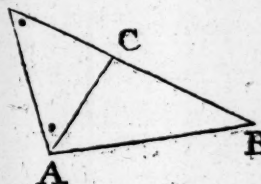
k Cor. 1.
to 10.

l 15. 1.

THEOREM XIX.

Any two sides (AC, BC) of a triangle (ABC) taken together, are greater than the third side (AB.)

In BC produced, let D there be taken $CD = CA$, and let AD be drawn. The angles D and DAC are equal m ; therefore BAD , which exceeds the latter n , must also exceed the former D ; and consequently BD (or $BC + AC$) must exceed AB o .



m 12. 1.
 n Ax. 2.

o Cor. to

C 3

THEO. 13. 1.

Elements of Geometry.

THEOREM XX.

Of all the right lines (PA, PB, PC) falling from a given point (P) upon an infinite right-line (RS), that (PA) is the least which is perpendicular to it; and, of the rest, that (PB) which is the nearest the perpendicular is less than any other (PC) at a greater distance.

For BAP being a right-
 P Hyp. angle $\angle$, ABP will be a-
 q Cor. 4. cute $\angle$, and therefore AP
 to 10. 1. $\perp$ BP.

r Cor. to
 13. 1.

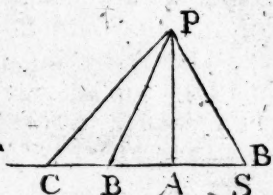
Again, when PB and R

PC are both on the same
 side of the perpendicular

r Cor. to
 9. 1.

PA; then is CBP $\perp$ right angle $\angle$ $\perp$ BCP $\angle$, and
 consequently PC $\perp$ PB.

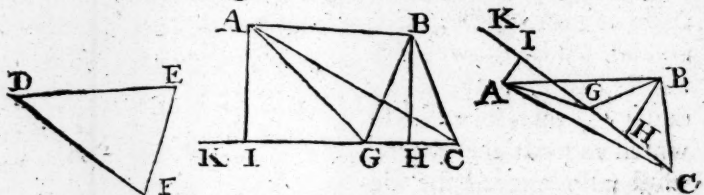
If PB be on the contrary side of the perpendicular
 to PC; from AC, let AB be taken $\perp$ AB; then
 r Ax. 10. the two lines PB, PB will be also equal $\perp$; and there-
 fore PC, which exceeds the one (by the preceding
 case) will also exceed the other.



THEOREM XXI.

Of two triangles (ABC, DEF) having two sides (AB, BC) of the one, equal to two sides (DE, EF) of the other, each to each respectively, the base of that (ABC) will be the greatest, which is subtended under the greatest angle.

Let the angle ABG = E, BG = EF (=BC) also



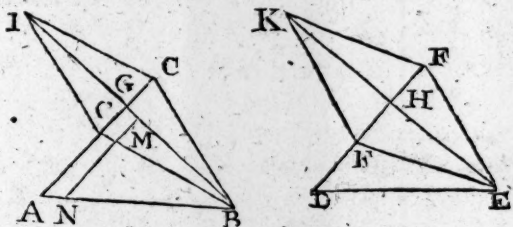
let AG and CG be drawn, upon the last of which,
 produced,

produced, let fall the perpendiculars BH and AI^a, ^a 4. 1.
 Since BG = BC^w, and, consequently, GH = HC^x, ^w Hyp.
 it is evident, that GI (whether the point I be confi- ^x 16. 1.
 dered as falling between G and K, or between G
 and H) will be less than CI^z; and therefore AG, ^z Ax. 2.
 or its equal DF^z, also less than AC^b. ^b Ax. 10.
^b 20. 1.

THEOREM XXII.

*Of two triangles (ABC, DEF), having one angle (BAC) in the one equal to one angle (EDF) in the other, and the sides (BC, EF) opposed to them also equal, that (ABC) will have the greatest base, where-
 of the opposite angle (ACB) differs the least from a
 right-angle.*

Let BG and EH be perpendicular to AC and DF, in which produced, take HK = HE, GI = GB, and BM = EH; also let MN be parallel to GA, meeting AB, produced if necessary, in N; and let CI and KF be drawn.



The angle ICG being = BCG^d, and the latter of
 these greater than EFH^e, (or KFH^a), thence is ^e Hyp.
 ICB = KFE; and consequently BI = EK^e; whence ^d Ax. 10.
 also BG ($\frac{1}{2}$ BI) = EH ($\frac{1}{2}$ EK) or its equal BM^d; and ^e 21. 1.
 therefore BA = BN, because AG and MN being
 parallels, both the points M and N will fall on the
 same side of AG. But BN (as the triangles NBM,
 DEH are equiangular, and have BM = EH^f) is ^f Hyp. and
 = DE^g: therefore BA is also greater than DE. ^g 7. 1.
^g 15. 1.

THEOREM XXIII.

If, of two triangles (ABC, ABD) standing upon the same base (AB), the one be wholly included within the other, the two sides (AD, BD) of the included one taken together, will be less, and the angle (D) contained by them greater, respectively, than the two sides (AC, BC), and the contained angle (C) of the other.

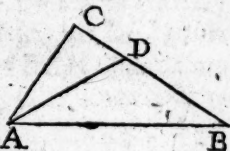
CASE I. If the vertex of the contained triangle be in one side of the other :

Then, AD is less than AC + CD¹; whence, by adding BD common, AC + BD will also

¹ Ax. 6. be less than AC + CD + BD¹, A

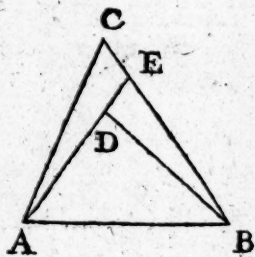
² Ax. 3. or than its equal AC + BC². B

¹ Cor. 9.1. But the angle ADB is $\sphericalangle$ ACB¹.



CASE II. If the vertex be within the other triangle.

Let AD be produced to meet BC in E: then (by the preceding case) the sum of AD and BD is less than the sum of AE and BE; which last sum, and consequently the former, is, again, less than the sum of AC and BC. Moreover, the angle ADB $\sphericalangle$ BED $\sphericalangle$ C.

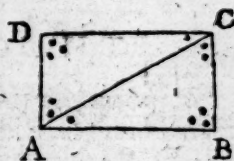


THEOREM XXIV.

The opposite sides (AB, DC) of any parallelogram (ABCD) are equal, as are also the opposite angles (B, D); and the diagonal (AC) divides the parallelogram into two equal parts.

For,

For, AB, DC, and AD, BC being parallels^a, the angle BAC is = DCA^b, and BCA = DAC^c, therefore the equiangular triangles ABC, ADC^d having AC common, are equal in all respects^e.



^a Def. 24.

^b 7. 1.

^c Cor. 1. to 10. 1.

^d 15. 1.

COROLLARY.

Hence, if one angle (B) of a parallelogram be a right angle, all the other three will be right ones: for D, being = B, is a right-angle; and BCD is = B, and DAB = D, by Theor. V.

THEOREM XXV.

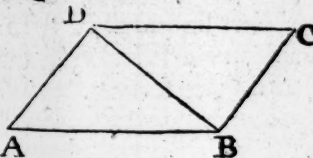
Every quadrilateral (ABCD) whose opposite sides are equal, is a parallelogram. (See the preceding scheme).

Let the diagonal AC be drawn; then the triangles ABC, ADC being mutually equilateral^a, they^b Hyp. will also be mutually equiangular^c; consequently^d 14. 1. AB will be parallel to DC, and AD to BC^e. 8. 1.

THEOREM XXVI.

The lines (AD, BC) joining the corresponding extremes of two equal, and parallel lines (AB, DC) are themselves equal and parallel.

Let the diagonal BD be drawn. Because AB and DC are parallel^a, the angle ABD is = CDB^b; ^a Hyp. therefore, BA being = DC^c, and BD common, the remaining sides and angles will likewise be respectively equal^d; and consequently AD parallel to BC^e.



^b 7. 1.

^c Hyp.

^d Ax. 10.

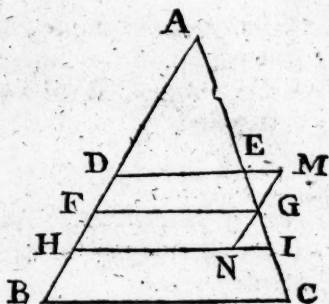
^e 8. 1.

THEO-

THEOREM XXVII.

If, in one side (AB) of a triangle (ABC), from three points (D, F, H) at equal distances (DF, FH), lines (DEM, FG, HI) be drawn parallel to the base, the parts (EG, GI) of the other side (AC) intercepted by them, will also be equal to each other.

Let NGM be parallel to AB, intersecting HI and DE in N and M. Then, the triangles IGN, MGE, having the angle $\text{IGN} = \text{EGM}^a$, $\text{ING} = \text{M}^b$, and $\text{GN} (= \text{FH}^c = \text{FD}^d) = \text{GM}^c$, will also have $\text{GI} = \text{GE}^e$.



COROLLARY I.

Hence it appears, that, if one side of a triangle be divided into any number of equal parts, and from the points of division lines be drawn parallel to the base, cutting the other side, they will also divide it into the same number of equal parts.

COROLLARY II.

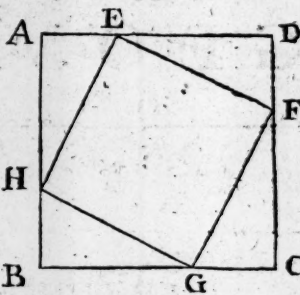
Hence, also, if two lines FG, HI, cutting the sides of a triangle, be parallel to each other, and another line DE be so drawn as to cut off $\text{FD} = \text{FH}$ and $\text{GE} = \text{GI}$, this line DE will be parallel to the two former.

THEO.

THEOREM XXVIII.

If in the sides of a square (ABCD), equally distant from the four angular points, there be taken four other points (E, F, G, H,) the figure (EFGH) formed by joining those points, shall also be a square.

For the wholes AD, DC, CB, BA being equal^f, and also the parts AE, DF, CG, BH^g, the remaining parts ED, FC, GB, HA must consequently be equal^h; whence, all the angles D, C, B, A being equalⁱ; B the sides EF, FG, GH,



^f Def. 26.

^g Hyp.

^h Ax. 5.

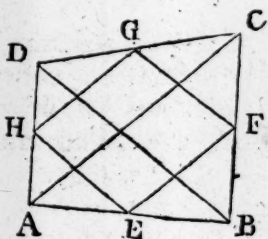
ⁱ Ax. 7.

HE will be equal likewise^k, and the angle DEF = ^k Ax. 10. AHE^k. Therefore, because DEH is = A + AHE^l, if from these, the equal angles DEF,¹ 9. 1. AHE be taken away, there will remain HEF = A^h = a right-angle^f. By the same argument (or by Theor. 25th, and the Corol. to the 24th) the other three angles will be right-angles.

THEOREM XXIX.

If all the sides of any quadrilateral (ABCD) be bisected, the figure (EFGH) formed by joining the points of bisection, will be a parallelogram.

Draw the diagonals AC and BD. Because EF and HG are parallel to ACⁿ, they are also parallel to each other^o. After the same manner is FG parallel to EH; therefore EFGH is a parallelogram^p.



ⁿ Cor. 1. to 27. 1.

^o 6. 1.

^p Def. 24.

The End of the FIRST BOOK,

E L E M E N T S

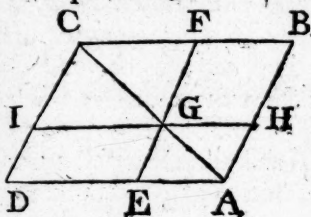
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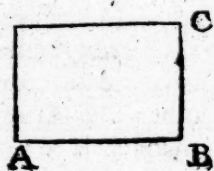
B O O K II.

D E F I N I T I O N S.

1. **I**N a parallelogram $ABCD$, if two right-lines EF , HI , parallel to the sides, intersecting the diagonal in the same point G , be drawn, dividing the parallelogram into four other parallelograms; those two GD , GB through which the diagonal does not pass, are called *Complements*; and the other two, HE , FI , parallelograms about the diagonal.



2. Every rectangle is said to be contained under the two right-lines AB , BC that are the base and altitude thereof.



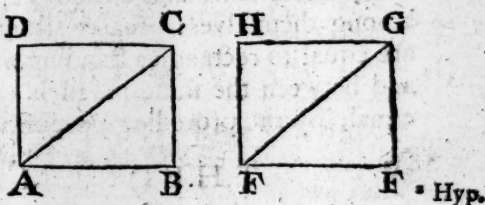
The

The rectangle contained under two right-lines AB and BC is often, for brevity sake, denoted by $AB \times BC$. But when the figure is a square, it is usually represented by placing the number 2 over the letter, or letters expressing the side thereof: thus AB^2 denotes the square made upon the line AB,

THEOREM I.

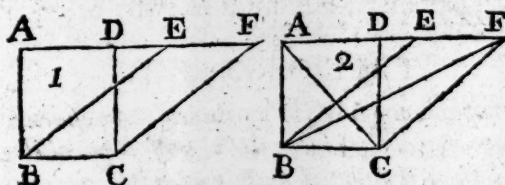
The rectangles (BD, FH) contained under equal lines, are equal.

For, let the diagonals AC, EG be drawn: then, because $AB = EF$, $BC = FG$, and $B = F^a$, the triangles ABC, EFG are equal^b. And, in the very same manner^b Ax. 10. will ADC and EHG appear to be equal. Therefore the whole rectangle ABCD is also equal to the whole rectangle EFGH^c.



THEOREM II.

Parallelograms (ABCD, BCFE) standing upon the same base (BC) and between the same parallel (BC, AF) are equal.



For, since (in Fig. 1.) the angle $F = BEA^d$, and $CDF = A^d$, the triangles FDC, EAB are equiangular^e; they are also equal^f, because $CF = BE^g$: therefore, if each be taken from the whole figure ABCF, there will remain $ABCD = EBCF^h$.

COROLLARY

COROLLARY I.

Hence, triangles BAC, BFC (*Fig. 2.*) standing upon the same base, and between the same parallels, are also equal, being the halves of their respective parallelograms ¹.

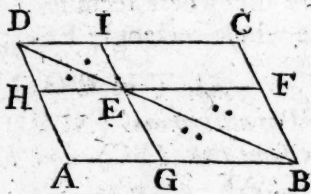
COROLLARY II.

Hence all parallelograms, or triangles, whatever, whose bases and altitudes are equal, are equal among themselves; because all such parallelograms are equal to rectangles standing on the same bases, and between the same parallels; and these last are equal, by the preceding proposition.

THEOREM III.

The complements (EC, EA) of any parallelogram (AC) are equal.

For, the whole triangle DCB being equal to the whole triangle DAB^k, and the parts DIE, EFB respectively equal to the parts DHE, EGB^k, the remaining parts EC, EA must likewise be equal¹.



THEOREM IV.

A trapezium (ABCD) whereof two sides (AD, BC) are parallel, is equal to half a parallelogram, whose base is the sum of those two sides, and its altitude, the perpendicular distance between them.

For,

For, in AD produced, take $DF = BC$; and let CG, DH and FE be all parallel to AB, meeting AF and BC produced, in G, H and E. Then AE



is a parallelogram of the same altitude with ABCD, having its base AF equal to the sum of AD and ^a BC : but this parallelogram, because $BG = HF$ ^o Cor. 2. and $CGD = CHD$ ^p 24. 1. is equally divided by the line ^q CD ; and so ABCD is the half thereof. ^q Ax. 4.

THEOREM V.

The sum of all the rectangles contained under a given line (AD), and all the parts (AH, HG, GB) of another (AB), any how divided, is equal to the rectangle contained under the two whole lines.

Let ABCD be the rectangle contained under the two whole lines, and let HF, GE be parallel to AD, meeting DC in F and E. Then will AF, HE, GC be rectangles ^r of the same



altitude with AC ^s; therefore $AF = AD \times AH$, $HE = AD \times HG$, and $GC = AD \times BG$ ^t; and ^u consequently $AD \times AB$ ($AC = AF + HE + GC$) ^t $= AD \times AH + AD \times HG + AD \times BG$ ^u.

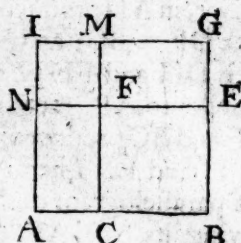
^r Cor. to 24. 1.
^s 24. 1. & Ax. 1.
^t 1. 2.
^u Ax. 3. & 4. 1.

THEOREM VI.

If a right-line (AB) be, any-wise, divided into two parts (AC, BC), the square of the whole line will be equal to the squares of both the parts, together with two rectangles under the same parts.

Let

Let ABGI be the square of AB, and CBEF that of BC, and let EF and CF be produced to meet the sides of the square ABGI in M and N.



From the equal quantities CM , EN take the equal quantities CF and EF , and there remains $\text{FM} = \text{FN}$; therefore, all the angles of the figure being right ones, NM is a square upon $\text{FN} (= \text{AC})$; and AF , FG are equal to two rectangles under BC and AC ; but $\text{AG} = \text{BF} + \text{FI} + \text{AF} + \text{FG}$, or $\text{AB}^2 = \text{BC}^2 + \text{AC}^2 + 2\text{AC} \times \text{BC}$.

COROLLARY I.

Hence, the square of any line is equal to four times the square of half that line.

COROLLARY II.

Hence, also, if two squares be equal, their sides must be equal; because unequal lines BA , BC have not equal squares.

THEOREM VII.

The difference of the squares (ABEH , ACIK) of any two unequal lines (AB , AC), is equal to a rectangle under the sum and difference of the same lines.

In EB , produced, take $\text{BF} = \text{AC}$; let FG be drawn parallel to EH , and let CI be produced both ways, to meet EH and FG in D and G . It is evident that DF is a rectangle, whose base $\text{GF} (= \text{CB})$ is the difference of the given lines AB , AC ; and whose altitude FE (because BE



^c Cor. 24.

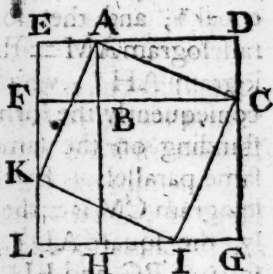
^d Ax. 24.1.

$= BA^2$, and $BF = AC^2$ is = the sum of the same ^{Def. 24.} lines: but this rectangle DF is $= DB + GB^2 =$ ^{Hyp.} $DB + DK$ (because $DK^h = GB$) = the square AE^2 ^{Ax. 3.} — the square AI . ^{1. 2.}

THEOREM VIII.

The square made upon the side (AC) subtending the right-angle of a plane triangle (ABC), is equal to both the squares (BE, BG) made upon the sides (AB, BC) containing that angle.

Let the sides of the squares BE , BG be produced to meet each other in L and D ; in which take KL and IG each equal to AE (or AB); and let CI , IK , and KA be drawn.



Since ABH and FBC (which are continued right-lines¹) are equal to each ^{2. 1.} other^k, EL , DG , ED , and LG will be all equal ^{Ax. 4.} among themselves¹; and so the angles E , D , G ^{24. 1.} and L being all right ones^m, $EDGL$ will be a ^{Hyp. &} square, and consequently $ACIK$ a square likewiseⁿ. ^{5. 1.} Now, if from the square DL , the four equal ^{28. 1.} trian- ^{Ax. 10.} gles ADC , CGI , ILK , and KEA be taken away, there will remain the square AI : and, if from the same DL , the two equal ^{1. 2.} parallelograms DB , BL ^{1. 2.} (which are equal to the said four triangles, because $DB =$ two of them¹) be taken away, then there will remain the two squares BE and BG . Consequently the square AI is = the two squares BE and BG . ^{Ax. 5.}

The same demonstrated otherwise.

Let AD be the square on the hypotenuse AC , and BG , BI the two squares on the sides AB and BC :

BC: let MBH be parallel to AE, meeting GF. (produced) in H; and let EA be produced to meet GH in N.

* Ax. 7. If from the equal ^r angles GAB, CAN, the angle NAB, common to both, be taken away, there will remain NAG

* Ax. 5. $\angle BAC$; whence, as the angle G is also $\angle ABC$, and the

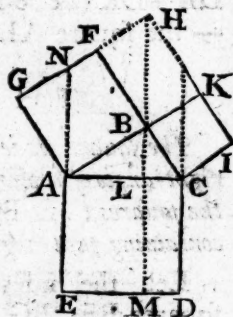
* Def. 26. side AG $= AB$, the sides AN and AC ($= AE$) are likewise equal; and therefore the parallelogram AM $=$ the paralle-

* 15. 1. logram AH; which last, and

* Cor. to 2. 2.

* 2. 2. consequently the former, is equal to the square BG^x standing on the same base AB, and between the same parallels. By the same argument, the parallelogram CM is $=$ the square BI: and, consequently, the square AD ($= AM + CM$) $=$ both the squares BG and BI.

* Ax. 4.



COROLLARY.

Hence, the square upon either of the sides including the right angle, is equal to the difference of the squares of the hypotenuse and the other side²; or, equal to a rectangle contained under the sum and difference of the hypotenuse and the other side².

* Ax. 5.

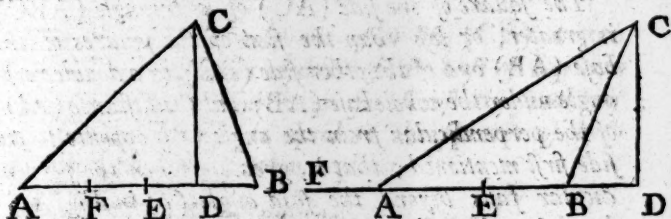
* 7. 2.

THEOREM IX.

The difference of the squares of the two sides (AC, BC) of any triangle (ABC) is equal to the difference of the squares of the two lines, or distances (AD, BD) included between the extremes of the base (AB) and the perpendicular (CD) of the triangle.

For, since $AC^2 = DC^2 + AD^2$ and $BC^2 = DC^2 + BD^2$ (by the precedent), it is evident that the difference

difference of AC^2 and BC^2 will be equal to the difference between $DC^2 + AD^2$ and $DC^2 + BD^2$, or b Ax. 5.



between AD^2 and BD^2 , by taking away DC^2 , c Ax. 6. common, from both.

COROLLARY I.

Since the rectangle under the sum and difference of any two unequal lines, is equal to the difference of their squares d , it follows, that the difference of a 7. 2. the squares (or the rectangle under the sum and difference) of the two sides of any triangle, is equal to the rectangle under the sum and difference of the distances included between the perpendicular and the two extremes of the base.

COROLLARY II.

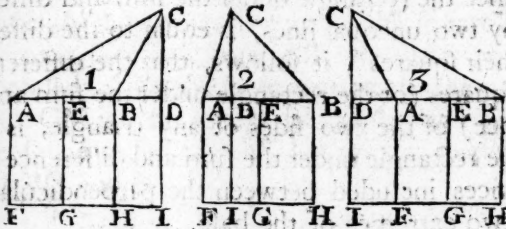
It follows, moreover, that the difference of the squares (or the rectangle under the sum and difference) of the two sides of a triangle, is equal to twice a rectangle under the whole base, and the distance of the perpendicular from the middle of the base.

For, let E be the middle of the base, and let EF be made $= ED$; then AF being $= BD$, the excess of AD above BD (or AF) will (in Fig. 1.) be $= DF = 2DE$; therefore the rectangle under the sum and difference of AD and BD ($= AC^2 - BC^2$) is $= AB \times 2DE$. Again (in Fig. 2.) AD + BD being $= AD + AF = FD = 2ED$, and b Ax. 4. $AD - BD = AB$, we have, also, in this case c , $AC^2 - BC^2 = AB \times 2DE$.

THEOREM X.

The square of one side (AC) of a triangle (ABC) is greater, or less than the sum of the squares of the base (AB) and of the other side (BC), by a double rectangle under the whole base (AB) and the distance (BD) of the perpendicular from the angle (B) opposite to the side first mentioned; that is, greater, when the perpendicular falls beyond the said angle (as in Fig. 1.); but less, when it falls on the contrary side (as in Fig. 2. and 3.)

Let the square ABHF, on the base AB, be divided into two equal¹ rectangles EF and EH by



the line EG, bisecting AB in E; and let the perpendicular CD be continued out to meet FH (produced) in I.

In Fig. 1. $AC^2 - BC^2 =$ twice the rectangle
¹ Cor. 2. $EI^1 = 2EH + 2BI^m = AH (AB^2) + 2BI$
^{to 9. 2.} $(2AB \times BD)$; therefore, if from the first and last
² Ax. 3. of these equal quantities, AB^2 be taken away, then
³ Ax. 5. AC^2 less both BC^2 and $AB^2 = 2AB \times BD^m$:

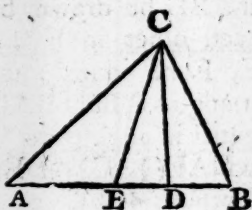
In Fig. 2. and 3. $BC^2 - AC^2 = 2EI^1 = 2BI -$
 $2BG^m = 2AB \times BD - AB^2$; and so, by adding
 AB^2 to the first and last of these equal quantities,
⁴ Ax. 4. we have here $AB^2 + BC^2 - AC^2 = 2AB \times BD^o$.

THEO-

THEOREM XI.

The double of the square of a line (CE) drawn from the vertex to the middle of the base of any triangle (ABC), together with double of the square of the semi-base (AE), is equal to the squares of both the sides (AC, BC) taken together.

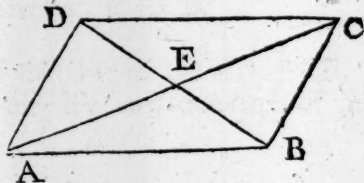
For, let CD be perpendicular to AB: then, because (by the precedent) AC^2 exceeds the sum of the two squares AE^2 and CE^2 (or BE^2 and CE^2) by the double rectangle $2AE \times ED$ (or $2BE \times ED$); and because BC^2 is less than the same sum by the same double rectangle; it is manifest that both AC^2 and BC^2 together must be equal to that sum twice taken; the excess on the one part making up the defect on the other.



THEOREM XII.

The two diagonals (AEC, BED) of a parallelogram (ABCD) bisect each other; and the sum of their squares is equal to the sum of the squares of all the four sides of the parallelogram.

For, the triangles AEB, DEC being equiangular^p, and having $AB = DC$ ^q, will also have $AE = CE$, and $BE = DE$ ^r. Moreover, be-



cause $2AE^2 + 2ED^2 = AD^2 + CD^2$, by taking^r the double of these, we have $4AE^2$ (AC^2) + $4ED^2$ (BD^2) = $AD^2 + BC^2 + CD^2 + AB^2$.

D 3

THE O-

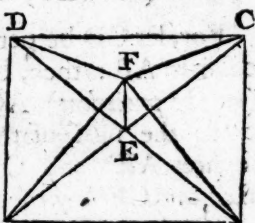
^p 3. and
^q 7. 1.
^r 24. 1.

^r 15. 1.
^q 11. 2.
^r Cor. 1.
to 6. 2.
^u Ax. 4.
and 24. 1.

THEOREM XIII.

If from any point (F), to the four angles of a rect-
angle (ABCD) four lines be drawn; the sums of the
squares of those drawn to the opposite angles will be
equal (I say, that $FA^2 + FC^2 = FB^2 + FD^2$).

For, let the diagonals AC and BD be drawn, bisecting
each other in E^x, and let E, F be joined; then the
triangles ABC, BAD being equal in all respects^w, thence
will $AE(\frac{1}{2}AC) = DE(\frac{1}{2}DB)$.
But $FA^2 + FC^2 = 2AE^2$
 $(2DE^2) + 2EF^2 = FB^2 + FD^2$.



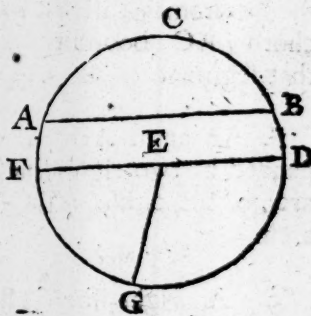
End of the SECOND BOOK.

ELEMENTS OF GEOMETRY.

BOOK · III.

DEFINITIONS.

1. **A**NY right-line FD, passing through E the center of a circle, and terminating in the circumference at both ends, is called a Diameter.



2. An arch of a circle, is any portion of the periphery, or circumference, as ACB.

3. The chord, or subtense of an arch ACB, is a right-line AB joining the two extremes of that arch.

4. A semi-circle is a figure contained under any diameter and either part of the circumference cut off by that diameter.

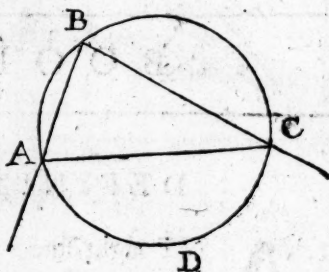
D 4

5. A

5. A segment of a circle is a figure contained under an arch ACB and its chord AB.

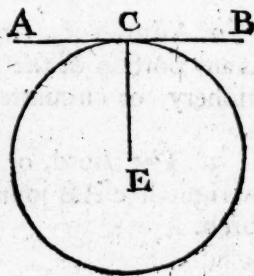
6. A Sector of a circle is a figure contained under two right-lines EF, EG, drawn from the center to the circumference, and the arch FG included betwixt them. When the two lines EF, EG, stand perpendicular to each other, then the Sector is called a quadrant.

7. An angle ABC is said to be in a segment of a circle ABC, when, being in the periphery thereof the right-lines BA, BC by which it is formed, pass through the extremes of the chord AC bounding that segment.

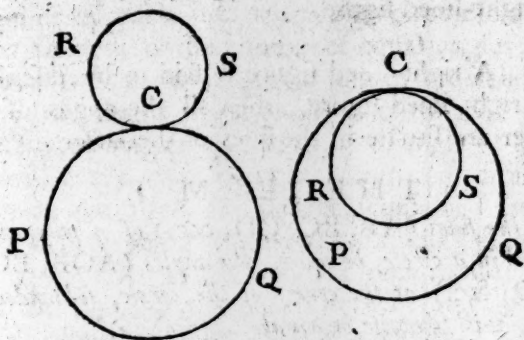


8. An angle ABC in the periphery, comprehended by two right-lines BA, BC, including an arch of the circle, ADC, is said to stand upon that arch.

9. A right-line AB is said to touch a circle, when, passing through a point (C) in the circumference thereof, it cutteth off no part of the circle.



10. Two circles (PCQ, RCS) are said to touch each other, when the circumferences of both pass



through one point (C) and yet do not cut each other.

11. Two circles, in the same plane, are said to cut one another, when they fall partly within, and partly without each other; or, when their circumferences cut each other.

12. A right-line is said to be applied, or inscribed in a circle, when both its extremes are in the periphery thereof.

13. A right-lined figure is said to be inscribed in a circle, when all its angles are in the circumference of the circle.

14. A circle is said to be described about a right-lined figure, when the periphery of the circle passes through all the angles of that figure.

15. A right-lined figure is said to be described about a circle, when all the sides thereof touch the circle.

16. A

16. A circle is said to be inscribed in a right-lined figure, when it is touched by all the sides of the right-lined figure.

17. A right lined figure is said to be inscribed in a right-lined figure, when all the angles of the former are situate in the sides of the latter.

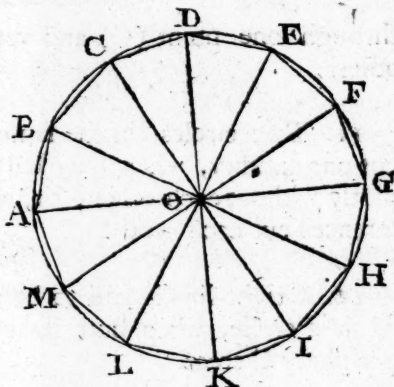
THEOREM I.

If the sides (AB, BC, CD, &c.) of a polygon inscribed in a circle, be equal, the angles (AOB, BOC, COD, &c.) at the center of the circle, subtended by them, will likewise be equal.

For AO, BO, CO, &c. being equal to each other^a, as well as AB, BC, CD &c. the triangles AOB, BOC, COD, are mutually equilateral; and therefore have all the angles AOB, BOC &c. equal to each other^b.

^a Def. 33. of 1.

^b 14. 1.



SCHOLIUM.

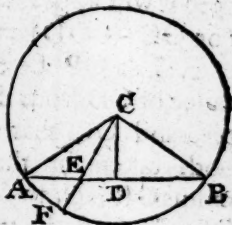
On this proposition depends the division of mathematical instruments for taking and measuring of angles. For, if, by repeated trials, or any other means, the circumference of a circle described about a center O, be divided into any number of parts AB, BC, CD &c. so that the chords be equal; then it is evident, from hence, that all the angles AOB, BOC, COD &c. which make up the four right-angles AOD, DOG, GOK, KOA at the center, will also be equal to each other, let the

the radius OA of the *instrument* be what it will.—
In the division of the circle for practical uses, the number of parts into which the circumference is thus divided, or the number of equal angles at the center, is 360; which equal angles are called *degrees*; so that a right-angle, consisting of 90 of these equal angles, is said to be an angle of 90 degrees; every angle being denominated, from the degrees and parts of a degree, contained therein; each degree being conceived to be subdivided into 60 equal parts, called *minutes*; each minute again into 60 equal parts, called *seconds*; and so on to thirds, fourths, fifths, &c. at pleasure.

THEOREM II.

Any chord (AB) of a circle, falls wholly within the same: and a perpendicular (CD) let fall thereon, from the center of the circle, will divide it into two equal parts.

Let C, A , and C, B be joined; and thro' any point E in the chord AB , let the right-line CEF be drawn, meeting the circumference in F .



It is evident, because $CA = CB^c$, that these equal lines are on different sides of the perpendicular CD^d ; and so, CE being $\supset CA$ or CF^d , $20. 1.$ the point E (take it where you will in the line AB) and consequently the line AB itself, will fall within the circle^e. Moreover, because the triangles $ACD, ^e Ax. 2.$ BCD have $CA = CB$ and CD common, thence will AD be also $= BD^f$. $16. 1.$

COROLLARY:

Hence a line bisecting any chord at right-angles, passes through the center of the circle.

THEO-

THEOREM III.

Any two chords (AB, DE) equally distant from the center (O) of a circle, are equal to each other.

Let the perpendiculars OF, OC be drawn, and let O, D and O, A be joined. Because OF = OC^e, OD = OA^f, and F and C are both right-angles^g, therefore is DF = AC^h, and consequently DE = 2DFⁱ = 2AC^k = AB^l.

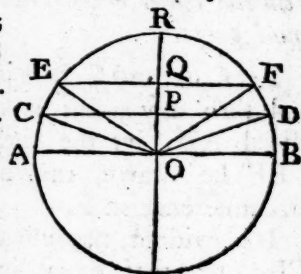


THEOREM IV.

In a circle (AEFB) the greatest line (AB) is the diameter; and, of all others terminating in the circumference, that (CD) which is nearest the center (O), is greater than any other (EF) further from it.

1. Draw OC and OD; then it will appear that AB (or OC + OD) = CD^m.

2. Let OP be the distance of CD from the center, and OQ that of EF, both taken in the same radius OR; Draw OE and OF. Because the triangles



Def. 33. DOC, OFE, have two sides equal each to eachⁿ, and have the contained angle DOC = the contained angle FOE^o; therefore, also, will the base DC be greater than the base FE^p; and consequently, greater than any other chord at the same distance, with EF^q.

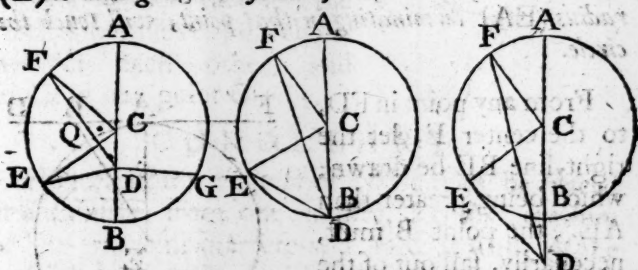
COROLLARY.

Hence, a right-line greater than the diameter, drawn from any point within a circle, will cut the circumference.

THEO-

THEOREM V.

If to the circumference of a circle (AFEB), from any point (D) which is not the center, right-lines (DA, DF, DE) be drawn, the greatest of all (DA) shall be that which passes through the center (C); and, of the rest, that (DF) whose other extreme (F) is placed nearest, in the circumference, to the extreme (A) of the greatest, will exceed any other (DE) whose extreme (E) is at a greater distance,



From the center C, let CE and CF be drawn.

1. $AD (= DC + CF) = DF$ Ax. 4.
2. Since DC is common $CF = CE$, and $\angle DCF$ 19. 1.
 $\angle DCE$, therefore is $DF = DE$. Ax. 2.

COROLLARY I.

Because no two lines, DE, DF, drawn from D, on the same side of the diameter AB, can be equal to each other^w, three equal right-lines cannot possibly be drawn from the periphery to any point, besides the center of the circle: and, therefore, if from a point in any circle, three equal right-lines can be drawn to the periphery, that point is the center of the circle.

COROLLARY II.

Hence it also follows, that no circle can be described to cut another FBG in more points than two: for, if it were possible to cut it in three points G, E, F, then right-lines drawn from the center

Q.

- ^a Def. 33. Q, to those points, would be all equal ^a, which is shewn to be impossible ^r, unless when the center
^{1.}
^r Cor. 1. Q coincides with C; and then the circles themselves
to 4. 3. will neither cut, nor touch, but coincide, and become one circle ^a.

THEOREM VI.

A right-line (FD) drawn through any point (A) in the circumference of a circle, at right-angles to the radius (EA) terminating in that point, will touch the circle.

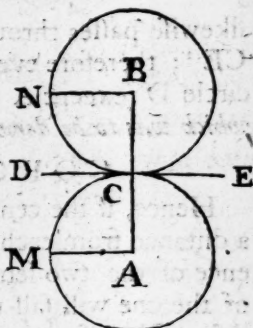
- From any point in FD, to the center E, let the right-line BE be drawn; which being greater than AE ^a, the point B must, necessarily, fall out of the
^a 20. 1.
^b Def. 33. circle ^b: and therefore, as the same argument holds good with regard to every other point in the line FD (except A), it is manifest that this line cuts off no part of the circle, but touches it, in one point only.



THEOREM VII.

If the distance (AB) of the centers of two circles, be equal to the sum of the two semi-diameters (AM, BN), the circles will touch each other, outwardly; and the right-line (AB) joining their centers, will pass through the point of contact.

In AB, take $AC = AM$, and let DCE be drawn perpendicular to AB: then, BC being also $= BN$, the circumferences of both circles will pass through the point C^d: but the right-line DE (by the precedent) falls wholly above the one, and wholly below the other; therefore the circles themselves fall wholly without each other, and touch in one point C only.



^c Constr. and Ax. 5.
^d Def. 33. 1.

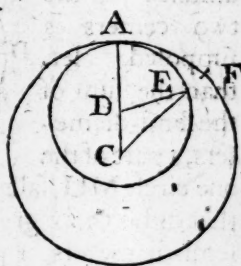
COROLLARY.

Hence, if the centers of two circles be placed at a distance, from one another, less than the sum of the two semi-diameters, a part, at least, of the one will be contained within the other: but, if the distance be greater than that sum, the two circles will then neither touch, nor cut each other.

THEOREM VIII.

If the distance (CD) of the centers of two circles (CAF, DAE) be equal to the difference of the two semi-diameters (CA, DE), then will those circles touch inwardly; and that radius (CA) of the greater, which is drawn through the center (D) of the lesser, will meet the two peripheries in the point of contact.

From any point E in the circumference of the lesser, to the two centers, let EC and ED be drawn. Because CA exceeds DE by the line DC^e, or because $DE + DC = CA^f$, $= DA + DC^g$, therefore is $DA = DE^h$; and so the circumference of the circle D



^e Hyp.
^f Ax. 4.
^g Ax. 3.
^h Ax. 5.

likewise

likewise passes through A : but CA is greater than CE¹ : therefore every point in the periphery of the circle D (except A only) falls within the circle C : *which was to be demonstrated.*

COROLLARY I.

Hence, if the centers of two circles be placed at a distance from each other, greater than the difference of the two semi-diameters, a part, at least, of the one will fall without the other ; but, if the distance be less than that difference, the lesser circle will then be contained wholly in the greater, but without touching it.

COROLLARY II.

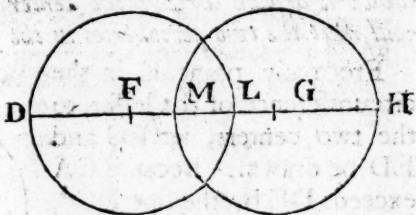
Hence, and from *the precedent*, it likewise appears, that if two circles touch, either inwardly or outwardly, a right-line, drawn through their two centers, will also pass through the point of contact : because they can only touch, when the distance of their centers is equal to the sum, or to the difference of their semi-diameters^k.

^k Cor. of
7. and
Cor. 1.
of 8.

THEOREM IX.

If the distance of the centers (F, G) of two circles (DL, MH) be less than the sum, and greater than the difference of the two semi-diameters (FL, GM), those circles will cut each other.

For, since the distance of the two centers is supposed less than the sum of the semi-diameters, a part of the

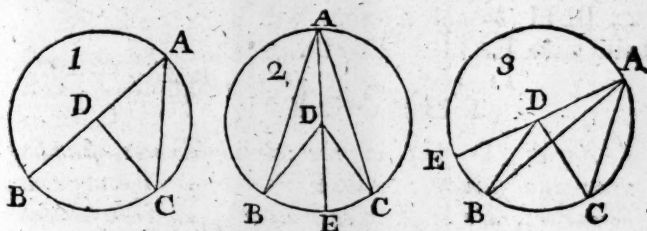


¹ Cor. to one circle MH, falls within the other DL¹ ; and since
7. 3. that distance is greater than the difference of those
^m Cor. to semi-diameters, a part of the same circle MH also
8. 3. falls without the circle DL^m : *which was to be provedⁿ.*
ⁿ Def. 11.
of 3.

THEO-

THEOREM X.

The angle (BDC) at the center of a circle, is double to the angle (BAC) at the circumference, when both angles stand upon the same arch (BC).



Let the diameter ADE be drawn.

In the first case (where AB passes through the center) $BDC = A + C^\circ = 2A^\circ$.

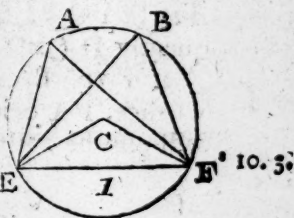
In the second case, $BDE = 2BAE$, (by case 1.)^{9. 1.} to which adding $CDE = 2CAE$, we have $BDC = 2BAC$.^{12. 1.}

In the third case, $CDE = 2CAE$ (by case 1.)^{9. Ax. 4.} from whence subtracting $BDE = 2BAE$, there remains $BDC = 2BAC$.^{1. Ax. 5.}

THEOREM XI.

All angles (EAF, EBF) in the same segment (EABF) of a circle, are equal to each other.

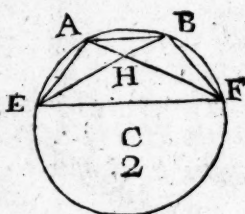
CASE I. If the segment be greater than a semi-circle; from the center C draw CE and CF; then EAF and EBF being each of them = to half ECF, they must necessarily be equal to each other.



E

CASE

CASE II. If the segment be less than a semi-circle; let H be the intersection of EB and AF: then the triangles AEH and BFH, having the angle AHE = BHF¹, and AEH = BFH (by case I.) they will also have EAH = FBH².

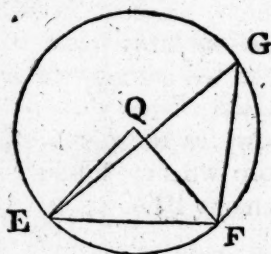
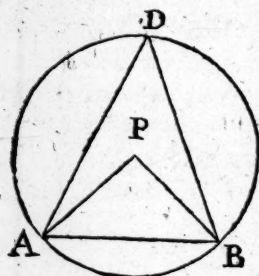


¹ 3. I.

² Cor. I.
to 10. I.

THEOREM XII.

Angles (D, G) in the circumferences, standing upon equal subtenses (AB, EF) of circles having equal diameters, are equal to each other. And the subtenses of equal angles, in the circumferences of circles having equal diameters, are also equal.



From the centers P, and Q, let PA, PB, QE, QF be drawn.

^w Hyp.

^x Def. 33.

of 14.

^y 14. I.

^z 10. 3.

^a Ax. 10.
of 1.

1. Hyp. Since $AB = EF^w$, and $AP = BP^x = EQ = FQ^w$; therefore is $P = Q^y$, and consequently $D (= \frac{1}{2} P^z = \frac{1}{2} Q) = G$.

2. Hyp. Because $D = G$, therefore $P = Q^z$; whence, PA being = QE, and PB = QF^w, AB will also be = EF^a.

COROLLARY.

Hence angles in the circumference, standing upon equal chords of the same circle, are equal.

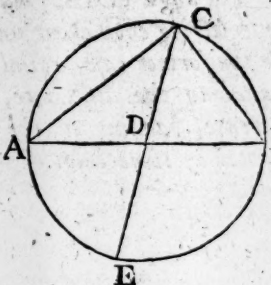
THEO.

THEOREM XIII.

The angle (ACB) in a semi-circle, is a right-angle.

Let the diameter CDE

be drawn.
Because ACD = half
ADE, and BCD = half
BDE^b, therefore is ACD
+ BCD (= ACB) = half
of ADE and BDE^c = half
two right-angles^d = one
right-angle.



B^b 10. 3.

^c Ax. 4. 1.

^d 1. 1.

THEOREM XIV.

The angle (CAB) included by a tangent to a circle and a chord (AC) drawn from the point of contact (A), is equal to the angle (AEC) in the alternate segment.

Let the diameter AOF be drawn, and E, F be joined.

The line DB falling wholly D
above the circle^e, OA is the
least line that can be drawn
to it from the center O^f; and
OAB is therefore a right-
angle^g: but FEA is also a
right-angle^h: therefore, if E
from these equal angles, the
equalⁱ angles FAC, FEC
(standing on the same arch FC) be taken away,^k Ax. 5.
there will remain BAC = AEC^k.



^e Def. 9.

3.

^f Def. 31.

1. and

^g Ax. 2.

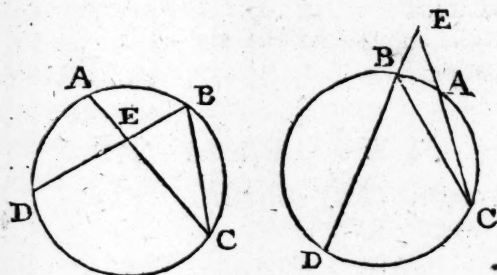
^h 20. 1.

ⁱ 13. 3.

^j 11. 3.

THEOREM XV.

The angle (DEC) made by two lines (DEB, CEA) intersecting each other within, or without a circle, is, in the former case, equal to the sum, and in the latter, equal to the difference, of two angles in the circumference, standing on the two arcs (DC, AB) intercepted by those lines.



Let the chord CB be drawn.

Then $DEC = DBC + ACB^e$, in the first case.
 And $DEC = DBC - ACB^h$, in the second case.

^e 9. 1.
^h 9. 1. and
 Ax. 5.

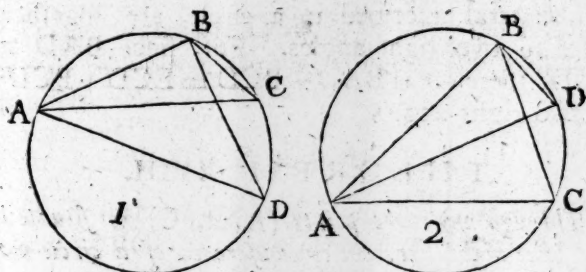
COROLLARY.

Hence an angle (E) formed below, or above the circumference of a circle, is greater, or less than an angle in the circumference, standing on the same arch.

THEOREM XVI.

The vertical angle (ABC) of any sblique-angled triangle (ACB) inscribed in a circle (ABCD) is greater, or less than a right-angle, by the angle (CAD) comprehended

prebended under the base (AC) and the diameter (AD) drawn from the extremity of the base.

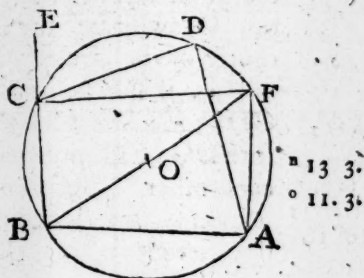


For, BD being drawn, ABD will be a right-angleⁱ, and $CAD = CBD$ ^k; therefore, in the first case, $ABC = \text{right-angle} + CAD$ ⁱ; and in the second, $ABC = \text{right-angle} - CAD$ ^m.
ⁱ 13. 3.
^k 11. 3.
ⁱ Ax. 4.
^m Ax. 5.

THEOREM XVII.

If any side (BC) of a quadrilateral (ABCD) inscribed in a circle, be produced out of the circle, the external angle (ECD) will be equal to the opposite, internal angle (BAD).

Let the diameter BF be drawn, and let AF and CF be joined: then the angle BAF being a right-angle ($= BCF$) $= ECF$ ⁿ, and DAF also $= DCF$ ^o, standing both on the same arch DF; thence will the remainders BAD and ECD be also equal^q.



ⁿ 13. 3.
^o 11. 3.

^q Ax. 5.

COROLLARY.

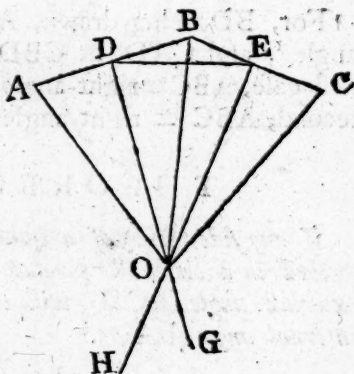
Hence the opposite angles BAD , BCD of any quadrilateral inscribed in a circle, are together, equal to two right-angles. For, since $BAD =$
^r Ax. 4. ECD , therefore is $BAD + BCD = ECD + BCD$
^s 1. 1. $=$ two right-angles ^s.

THEOREM XVIII.

Through any three points (A, B, C) not situate in the same right-line, the circumference of a circle may be described.

Draw AB and BC , which let be bisected by the perpendiculars DG and EH , infinitely produced on that side of AB or BC , on which the angle ABC is formed.

These perpendiculars, I say, will intersect each other; and the point of intersection O , will be the center of the circle.



For, if DE be drawn, it is plain, that the angles GDE , HED are less than two right-angles^r; therefore DG , EH , not being parallels^u, they will meet each other^w. Hence, if from the point of intersection O , the right lines OA , OB , OC be drawn, the triangles ADO , BDO , having two sides equal, each to each^x, and the angles ADO , BDO , contained by them, equal^y, will likewise have $AO = BO$ ^z. After the very same manner is $CO = BO$; therefore $AO = BO = CO$ ^a: whence, the circumference of a circle described from the center O , at
the

^r Ax. 2.

^u Cor. 2.

to 7. 1.

^w Def. 13.

of 1.

^x Constr.

^y Ax. 7.

^z Ax. 10.

^a Ax. 1.

the distance of AO, will also pass through B and C^b. ^b Def. 33. 1.

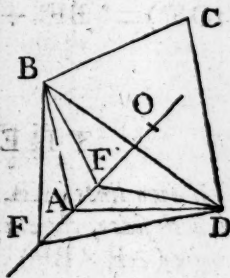
SCHOLIUM.

Hence the method of describing the circumference of a circle through three given points, is manifest.

THEOREM XIX.

If the opposite angles (BAD, BCD) of a quadrilateral (ABCD) be equal to two right-angles, a circle may be described about that quadrilateral.

For the circumference of a circle may be described thro' any three points B, C, D, (*by the precedent.*) But, if you deny that it passes thro' A; then, thro' the center O, let OAF be drawn, and let it (if possible) pass through some other point F in the line OAF, (for it must cut this line somewhere^c); also let

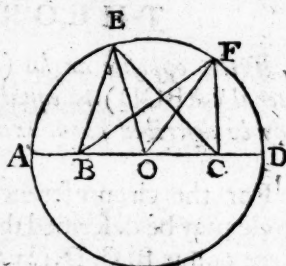


BF and DF be drawn. Because $BFD + BCD =$ ^c Cor. to 4. 3. two right-angles^d $= BAD + BCD^e$; therefore ^d Cor. to 17. 3. must $BFD = BAD^f$: which is impossible^g. There- ^e Hyp. fore the circumference of the circle described ^f Ax 5. through B, C and D, must also pass through A. ^g 23. 1.

THEOREM XX.

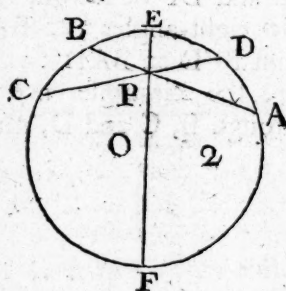
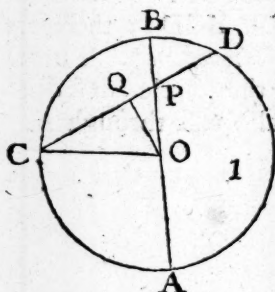
If from two points (B, C) in the same diameter (AD), equally distant from the center (O) of a circle, right-lines (BE, CE; BF, CF) be drawn to meet, two by two, in the circumference; the sum of the squares of any two corresponding ones, will be equal to the sum of the squares of any other two, meeting in like manner.

For, if OE and OF be drawn; then will $BE^2 + CE^2 = {}^h 2BO^2 + 2OE^2$
 11. 2. $(2OF^2) = {}^h BF^2 + CF^2$.



THEOREM XXI.

If two lines (AB, CD), terminated by the periphery on both sides, cut each other within a circle, the rectangle (AP × BP) contained under the parts of the one, will be equal to the rectangle (CP × DP) contained under the parts of the other.



CASE I. If one of the two lines (AB) passes through the center O; then let OQ be drawn perpendicular to the other CD, and let OC be joined,

It

It is plain, because $QD = QC$ ¹, that DP is ^{i 2. 3.} equal to the difference of the segments CQ and PQ ^k: but the rectangle under the sum and difference of the two sides OC, OP , of any triangle COP , is equal to the rectangle under the whole base CP , and the difference of its two segments¹; therefore, the ^{1 Cor. 1.} sum of the two sides OC, OP being $(= OA + OP)$ ^{to 9. 2.} $= AP$, and their difference $(= OB - OP) = BP$, thence is the rectangle contained under AP and BP equal to the rectangle contained under CP and DP .

CASE II. If neither of the two lines pass through the center; let the diameter EPF be drawn; then, by the preceding case, $AP \times BP = FP \times EP = CP \times DP$.

THEOREM XXII.

If from two points (A, C) in the circumference of a circle, two lines (AP, CP) be drawn, to pass through and meet without the circle; the rectangle ($AP \times BP$) contained under the whole and the external part of the one, will be equal to the rectangle ($CP \times DP$) contained under the whole and the external part of the other.

Through the center O , let PF be drawn, meeting the circumference in E and F ; let OQ be perpendicular to AP , and let A, O be joined.

Then, (by *Cor. 1. to 9. 2.*) the rectangle contained under PF $(= PO + OA)$ and PE $(= PO - OA)$ is = the rectangle contained under AP and PB . After the very same manner $PF \times PE = CP \times DP$: therefore $AP \times BP = CP \times DP$ ^m.



COROL. ^m Ax. 1.

COROLLARY I.

Hence, if PS be a tangent at S, and the radius OS be drawn; then, PF being = the sum of PO and OS, and PE = their difference; it follows, that

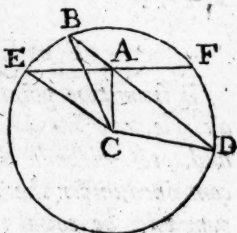
Cor. to 3. 2. $PS^2 = PF \times PE = PC \times PD.$

THEOREM XXIII.

If from the center (C) of a circle, to a point (A) in any chord (BD), a line (CA) be drawn; the square of that line, together with the rectangle contained under the two parts of the chord, will be equal to a square made upon the radius of the circle.

Let EAF be another chord, perpendicular to CA, and let C, E be joined.

2. 3. Since $AF = AE^2$, thence
 1. 2. will $AE^2 = AE \times AF =$
 21. 3. $AB \times AD$; to which equal
 3. 2. quantities adding AC^2 , we have
 $CE^2 = AB \times AD + AC^2.$



COROLLARY.

Hence the square of a line (AC) drawn from any point in the base of an isosceles triangle (BCD) to the opposite angle, together with the rectangle of the parts of the base, is equal to a square made upon one of the equal sides of the triangle.

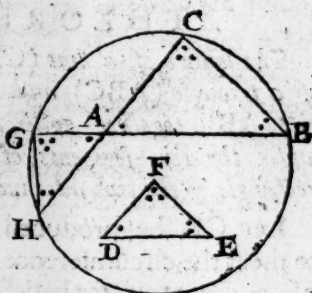
THEOREM XXIV.

The rectangles contained under the corresponding sides of the equiangular triangles (ABC, DEF) taken alternately, are equal.

I say, if $A = D$, $B = E$ and $C = F$, then
 will $AB \times DF = AC \times DE.$

In

In BA produced, let AG be taken = DF; let GCB be the circumference of a circle passing through the three points B, C, G, meeting CA produced in H; and let G, H be joined.



^v 18. 3.

Because the angle H = B' = E', HAG =

^a BAC = D', and AG = DF', thence is AH ^z Hyp. = DE ^b; and therefore AC × DE = AC × AH ^c 15. 1. = AB × AG ^d = AB × DF ^c.

^y 11. 3.

^z Hyp.

^a 3. 1.

^b 15. 1.

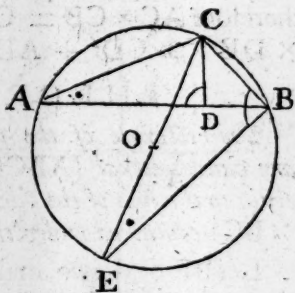
^c 1. 2.

^d 21. 3.

THEOREM XXV.

The rectangle under the two sides (AC, BC) of any triangle (ABC) is equal to the rectangle under the perpendicular (CD) to the base thereof, and the diameter (CE) of the circumscribing circle.

For, B, E being joined, the angles A, E will be equal; and ADC, EBC both right-angles^e; and, consequently the triangles ACD, ECB equiangular^h; therefore AC, EC; CD, CB being corresponding sides, opposed to equal angles, the rectangle AC × CB, contained under the first and last of them, will be equal to the rectangle EC × CD contained under the other twoⁱ.



^f 11. 3.

^g Hyp.

and 13.

^h 3.

ⁱ Cor. 1.

to 10. 1.

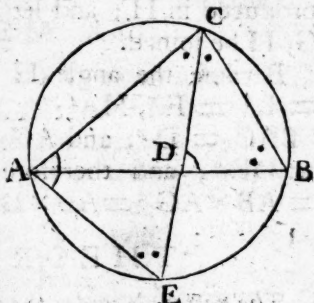
¹ 24. 3.

THEO.

THEOREM XXVI.

The square of a line (CD) bisecting any angle (C) of a triangle (ABC) and terminating in the opposite side (AB), together with the rectangle (AD $\times$ BD) under the two segments of that side, is equal to the rectangle of the two sides including the proposed angle.

Let CD be produced to meet the circumference of a circle, described ^k through the points A, C, B, in E; and let AE be drawn.



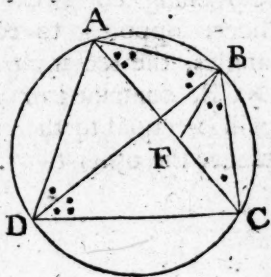
The angles E and B, standing upon the same segment AC, are equal ^l; and ACE is equal DCB (by hypothesis); therefore the triangles AEC, DCB are equiangular ^m; whereof AC, CD; CE, CB are corresponding sides, opposed to equal angles: therefore $AC \times CB = CD \times CE$ ⁿ $= CD^2 + CD \times DE$ ^o $= CD^2 + AD \times DB$ ^p.

^q 11. 3.
^r Cor. 1.
to 10. 1.

THEOREM XXVII.

The rectangle of the two diagonals (AC, BD) of any quadrilateral (ABCD) inscribed in a circle, is equal to the sum of the two rectangles (AB $\times$ DC, AD $\times$ BC) contained under the opposite sides.

Let BF be drawn, making the angle CBF = ABD, and meeting AC in F.



Because the angle BCF = ABD ^q, and CBF = ABD ^r, the triangles CBF, DBA are equiangular ^s; and therefore BC, BD; CF, AD, being corresponding sides, the rectangles $BC \times AD$, and $BD \times$

^t 11. 3.
^u Constr.
^v Cor. 1.
to 10. 1.

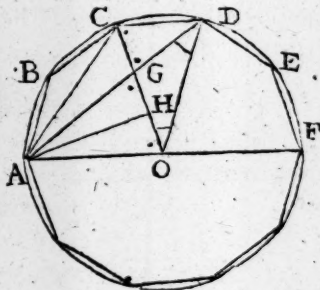
CF

CF will be equal¹. Again, the angle ABF being^t 24. 3.
 $\angle CBD^u$, and $\angle BAF = \angle BDC^w$; the triangles ABF^u Ax. 4. 1.
 and BDC, are, likewise, equiangular; and conse-^w 11. 3.
 quently, AB, BD; AF, DC being corresponding
 sides, $AB \times DC = BD \times AF^t$; to which adding
 $BC \times AD = BD \times CF$ (so proved above) we have
 also $AB \times DC + BC \times AD = BD \times AF + BD \times$
 $CF = BD \times AC^z$. 5. 2.

THEOREM XXVIII.

If the radius of a circle (OADF) be so divided
 into two parts, that the rectangle under the whole
 and the one part shall be equal to the square of the other
 part; then will this last part be equal to the side (CD)
 of a regular decagon (ABCDEF, &c.) inscribed in
 the circle; and that line whose square is equal to the
 two squares, of the whole and of the same part, will
 be equal to the side (AC) of a regular pentagon in-
 scribed in the same circle.

Draw the radii OA,
 OC, OD, OF; also draw
 AD, cutting OC in G,
 and let AH be perpen-
 dicular to OG.



The triangle ODG,
 having the angle COD
 $(= \frac{1}{2} \text{DOF}^y = \text{OAD}^z)$
 $= \text{ODA}^a$, is isosceles^b:
 moreover the triangle
 AOG, having $\text{AGO} (= \text{GDO} + \text{DOG}^c =$
 $2\text{DOC}^a) = \text{AOC}$, is likewise isosceles^d; as is also^d 18. 1.
 the triangle CDG, because, $\text{CGD} = \text{AGO}^c$, ^c 3. 1.
 and $\text{CDG} (\text{CDA}) = \text{FAD}^f$, the triangles AOG,^f Cor. to
 CDG are equiangular. Therefore, CD, AO; CG,^{12. 3.}
 GO being corresponding sides, we have $\text{CG} \times \text{AO}$
 $(\text{CG} \times \text{CO}) = \text{CD} \times \text{GO}^e = \text{GO}^2$, because GO^e 24. 3.
= GD

y 1. 3.
 z 10. 3.
 a 12. 1.
 b 18. 1.

^h 18. 1. $=GD = DC^h$: whence the former part of the proposition is manifest.

¹ 16. 1. Again, because $AG=AO$, HG will be $=HO^1$; and so GC being the difference of the segments HO and HC , we have (by *Cor. 1. to 9. 2.*) $AC^2 - AO^2 = CO \times CG = OG^2$ (as above); and consequently $AC^2 = AO^2 + OG^2$.

End of the THIRD BOOK.

ELEMENTS

OF

GEOMETRY.

BOOK IV.

DEFINITIONS.

1. **R**ATIO is the proportion which one magnitude bears to another magnitude of the same kind, with respect to quantity.

The measure, or quantity of a ratio is conceived by considering what part, or parts the magnitude referred, called the antecedent, is of the other, to which it is referred, called the consequent.

2. Three quantities, or magnitudes A, B, C, are said to be proportional; when the ratio of the first A to the second B, is the same as the ratio of the second B, to the third C.

3. Four quantities A, B, C, D, are said to be proportional, when the ratio of the first A to the second B, is the same as the ratio of the third C to the fourth D.

To

To denote that four quantities A, B, C, D are proportional, they are usually wrote thus, $A : B :: C : D$; and read thus, as A is to B, so is C to D. But when three quantities A, B, C are proportional, the middle one is repeated, and they are wrote thus, $A : B :: B : C$.

4. Of three proportional quantities, the middle one is said to be a Mean-proportional between the other two; and the last, a Third-proportional to the first and second.

5. Of four proportional quantities, the last is said to be a Fourth-proportional to the other three, taken in order.

A, B, C,
D, E.
1. 2. 4. 8.
16.

6. Quantities are said to be continually proportional (or in continual proportion) when the first is to the second, as the second to the third, as the third to the fourth, as the fourth to the fifth, and so on.

7. In a series, or rank of quantities continually proportional, the ratio of the first and third, is said to be duplicate to that of the first and second; and the ratio of the first and fourth, triplicate to that of the first and second.

8. Any number of quantities, A, B, C, D being given, or propounded, the ratio of the first (A) to the last (D) is said to be compounded of the ratios of the first to the second, of the second to the third, and so on to the last.

9. Ratio of equality, is that which equal quantities bear to each other.

*It may be observed here, that ratio of equality, and equality of ratios, are, by no means, synonymous terms :
since*

since two or more ratios may be equal, though the quantities compared are all unequal. Thus, the ratio of 2 to 1, is equal to the ratio of 6 to 3, (2 being the double of 1, and 6 the double of 3); yet none of the four numbers are equal.

10. Inverse ratio is, when the antecedent is made the consequent, and the consequent the antecedent. Thus, if $2 : 1 :: 6 : 3$; then, inversely, $1 : 2 :: 3 : 6$.

11. Alternate proportion is, when antecedent is compared with antecedent, and consequent with consequent.

As, if $2 : 1 :: 6 : 3$; then, by alternation (or permutation) it will be $2 : 6 :: 1 : 3$.

12. Compound ratio is, when the antecedent and consequent, taken as one quantity, are compared either with the consequent, or with the antecedent.

Thus, if $2 : 1 :: 6 : 3$; then, by composition, $2 + 1 : 1 :: 6 + 3 : 3$, and $2 + 1 : 2 :: 6 + 3 : 6$.

13. Divided ratio is, when the difference of the antecedent and consequent is compared, either with the consequent, or with the antecedent.

Thus, if $3 : 1 :: 12 : 4$; then, by division, $3 - 1 : 1 :: 12 - 4 : 4$, and $3 - 1 : 3 :: 12 - 4 : 12$.

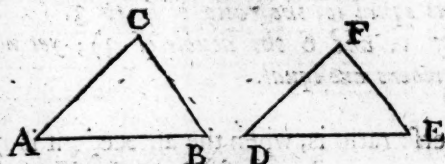
These four last definitions, which explain the names given by Geometers to the different ways of managing and diversifying of proportions, are put down here for the sake of order; but are not to be used, or referred to, in any shape, till those properties and relations are demonstrated; which is effected in the three first Theorems of this book.

14. Similar (or like) right-lined figures are such, which have all their angles equal, one to another

F

respec-

respectively, and also the sides about the equal angles proportional.



Thus, if the angle $A = D$, $B = E$, $C = F$; also $AC : AB :: DF : DE$, $BA : BC :: ED : EF$, &c. then the figures ABC , DEF are said to be *similar*.

A X I O M S.

1. The same quantity being compared with ever so many equal quantities, successively will have the same ratio to them all.

2. Equal quantities, have to one and the same quantity, the same ratio.

3. Quantities having the same ratio to one and the same quantity, or to equal quantities, are equal among themselves.

4. Quantities, to which one and the same quantity has the same ratio, are equal.

5. If two quantities be referred to a third, that which is the greatest will have the greatest ratio.

6. If two quantities be referred to a third, that is the greatest which has the greatest ratio.

7. Ratios, equal to one and the same ratio, are also equal, one to the other.

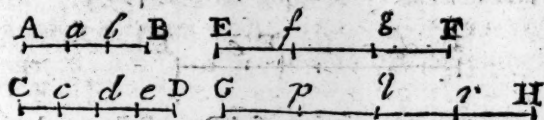
8. If two quantities be divided into, or composed of parts, that are equal among themselves, or all of the same magnitude; then will the whole of the one, have the same ratio to the whole of the other, as the number of the parts in the one, has to the number of equal parts in the other.

9. If the double, treble, or quadruple, &c. of every part of any quantity be taken, the aggregate will

will be the double, treble, or quadruple, &c. of the whole quantity propounded.

THEOREM I.

Equimultiples of any two quantities (AB, CD) are in the same ratio as the quantities themselves.



Let the ratio of AB to CD be that of any one number M (3) to any other number N (4), or, which is the same, let AB contain M (3) such equal parts (Aa, ab, bB), whereof CD contains the number N (4). Let there be taken Ef, fg, gF any equimultiples of Aa, ab, bB, respectively; and let Gp, pq, qr, rH be the same multiples of Cc, cd, de, eD: so shall the whole EF be the same assigned multiple of the whole AB, and the whole GH of the whole CD, as each part in the one, is of its correspondent in the other^a. And, since the parts Aa, ab, bB, Cc, cd, &c. are all equal^b; their equimultiples Ef, fg, gF, Gp, pq, &c. will also be equal^c. Therefore EF is in proportion to GH, as the number of the parts in EF is to the number of equal parts in GH^d, or (which is the same) as the number of parts in AB to the number of parts in CD, that is, as AB is to CD^e: which was to be demonstrated.

COROLLARY.

Hence, like parts of quantities, have the same ratio as the wholes; because the wholes are equimultiples of the like parts.

THEOREM II.

The two antecedents (AB, DE) of four proportional quantities, of the same kind (AB, BC, DE, EF) are in the same ratio with the two consequents (BC, EF).



Let the common ratio of AB to BC, and of DE to EF, be that of any one number $M(5)$ to any other number $N(3)$; then will AB contain^a $M(5)$ such equal parts (Aa) whereof BC contains^c $N(3)$; and DE will, in like manner, contain $M(5)$ such equal parts (Dd), whereof EF contains $N(3)$. And so, AB and DE, as well as BC and EF, being equimultiples of Aa and Dd , thence will AB : DE :: $Aa(Bb) : Dd(Ee)$:: BC : EF^f.

^a Ax. 8.
^c of 4.
^f 1. 4.

COROLLARY.

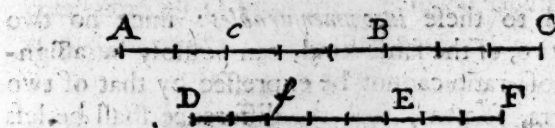
That the proportionality will subsist, when the consequents are taken as antecedents, and the antecedents as consequents, also appears from hence. For BC : AB :: number of parts in BC (or EF) : number of parts in AB (or DE) :: EF : DE^c.

THEOREM III.

Of four proportional quantities (AB, BC, DE, EF) the sum, or difference of the first antecedent and consequent ($AB \pm BC$) is to the first antecedent, or consequent, as the sum, or difference of the second antecedent and consequent ($DE \pm EF$) is to the second antecedent, or consequent.

Let

Let what was premised in the demonstration of the preceding theorem, be retained here : then will



AC ($AB+BC$) be in proportion to AB, as the number of parts in AC is to the number of equal parts in AB^s, or as the number of parts in DF^s Ax. 8.4. ($DE+EF$) to the number of equal parts in DE, that is, as DF ($DE+EF$) is to DE^s. Again, if from AB and DE, be taken away $Bc=BC$, and $Ef=EF$, then will the difference Ac be in proportion to AB, as the number of parts in Ac (or Df) is to the number of parts in AB (or DE^s), that is, as Df is to DE. In the same manner it will appear, that $AB+BC : BC :: DE+EF : EF$; and $AB-BC : BC :: DE-EF : EF$.

COROLLARY.

It will appear from hence, that the sum of the greatest and least ($AB+EF$) of four proportional quantities (of the same kind) will exceed the sum ($BC+DE$) of the two means : because, AB being supposed greater than DE, Ac will be greater than Df , in the same proportion^b : and, if to these^h 3. 4. there be added $BC+EF$ (common;) then will the sum $Ac+BC+EF$ ($AB+EF$) be also greater than the sum $Df+BC+EF$ ($DE+BC$.)ⁱ Ax. 6.1.

SCHOLIUM.

In the demonstration of this, and the preceding theorems, the antecedent and consequent are supposed to be divided into parts, all mutually equal among themselves. But it is known to Mathematicians, that there are certain quantities, or mag-

nitudes that cannot possibly be divided in that manner, by means of a *common measure*.²⁵ The Theorems themselves are, nevertheless, true, when applied to these *incommensurables*: since no two quantities, of the same kind, can possibly be assigned, whose ratio cannot be expressed by that of two numbers, so near, that the difference shall be less than the least thing that can be named. But if the matter, viewed in this light, should not appear sufficiently *scientific*, and you will not (in the preceding theorem) allow the ratio of AC to BC, to be *exactly* the same with that of DE to EF, when AB, BC, and DE, EF, are incommensurables; then let it, if possible, be as some quantity *aC* (less than AC) is to BC, so is DE to EF. Let *Bb* be a part or measure of BC less than the difference (*Aa*) between AB and *aB*; let *Bp* be that multiple of *Bb*, which least exceeds *Ba*, and let *qE* be to EF, as *pB* to BC.

It is evident, that $\frac{Aa}{pB} = \frac{Bb}{BC}$

pB is less than AB $\frac{Dq}{EF}$

(because $aB \supset Bb \supset Dq$)

* Hyp. ^{aA}; and that *qE* is

also less than DE,

because the ratio of *qE* to EF, being equal to that

of *pB* to BC¹, it must necessarily be less than that

¹ Ax. 5. 4. of AB to BC¹, or of DE to EF, and so *qE* less

² Ax. 6. 4. than DE².

Now, if (as is supposed) the ratio of *aC* to BC

can be = the ratio of DE to EF, it must, of consequence, be greater than the ratio of *qE* to EF³, or

(which is the same) than the ratio of *pC* to BC¹,

which is impossible. In like manner it will appear,

that no quantity, that is greater than AC, can possibly be to BC, as DE is to EF. Therefore AC:BC

: : DE : EF. And by the same kind of argumentation (authorized and adopted by Euclid himself,

in

³ Ax. 5. and 7. 4. ⁴ Proof of 3. 4.

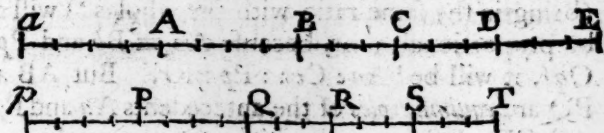
in his twelfth book) any difficulties, or scruples, that may be elsewhere brought, from the incommensurability of quantities, may be obviated and removed.

THEOREM IV.

If, of four proportionals (AB, BC, PQ, QR) equimultiples of the antecedents (AB, BC) be taken, and compared with any equimultiples of their respective consequents (BC, QR), the ratios will be the same, and the four quantities proportionals.

Let the common ratio of AB to BC, and of PQ to QR, be that of any one number M to any other number N : so shall AB contain M such equal parts whereof BC contains N , and PQ, in like manner, M such equal parts whereof QR contains N . Ax. 8.4.

Let CD, DE be taken each = BC, and RS, ST each = QR, so that BE and QT may be equimultiples of BC and QR; and let CD, DE; RS, ST be conceived to be divided, each into the same number of parts with BC, or QR. In like man-

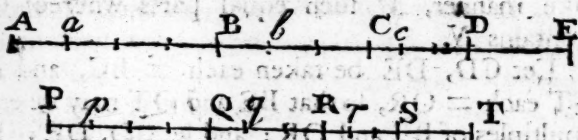


ner, let aB and pQ be taken as equimultiples of AB and PQ, &c. Then will the number of parts in BD = number of parts in QS , and the number of parts in BE = number of parts in QT . Ax. 4.1. and so likewise with respect to aB and pQ . Therefore aB is to BE , as the number of parts in aB to the number of (equal) parts in BE , or, which is the same thing, as the number of parts in pQ to the number of parts in QT , that is, as pQ is to QT , which was to be demonstrated.

THEOREM V.

If of two ranks of quantities (AB, BC, CD; PQ, QR, RS), the ratio of the first and second, in the one, be equal to the ratio of the first and second in the other, and the ratio of the second and third, in the one, equal likewise to the ratio of the second and third in the other; then, also, shall the ratio of the first to the third, be the same in the one rank, as in the other.

Let the common ratio of AB to BC, and of PQ to QR, be still expressed as in the preceding demonstrations. Let, moreover, CD and RS be conceived to be divided, each into the same number of parts with BC and QR.



Because the quantities BC, CD, QR, RS, are proportional*, their like parts Bb, Cc, Qq, Rr (being in the same ratio with the wholes*) will also be proportionals; or, because Aa = Bb and Pp = Qq, it will be Aa : Cc :: Pp : Rr. But AB and PQ are equimultiples of the antecedents Aa and Pp; and CD, RS are equimultiples of the consequents Cc and Rr; therefore AB : CD :: PQ : RS; which was to be demonstrated.

* When in two ranks of quantities, the proportions are inordinate, as AB : BC :: QR : RS, and BC : CD :: PQ : QR; the same thing may be demonstrated; and that in the very same manner, except only, that QR must here be divided into the same number of parts with AB, and PQ.

COROL.

COROLLARY I.

If other quantities DE, ST be taken, still proportional to the two next preceding them, so that $CD : DE :: RS : ST$; then, by the same argument (regard being had to AB, CD, DE in the one rank, and PQ, RS, ST in the other) it is evident, that $AB : DE :: PQ : ST$. And thus we may go on, still affixing other quantities, as many as we please; and the ratio of the first and last, will always be the same in one rank, as in the other. Therefore ratios compounded of the same number of like, or equal ratios, are equal. Def. 8.
of 4.

COROLLARY II.

It is also evident from hence, that if any two quantities be taken proportional to the two consequents of an assigned proportion, they will also be proportionals when compared with antecedents; and *vice versa*. For, the two quantities CD and RS, when compared, successively, with the consequents, and antecedents of the given proportion $AB : BC :: PQ : QR$, appear to be proportional, in the one case, as well as in the other. 5. 4.

THEOREM VI.

If, to the two consequents (BC, KL) of four proportionals (AB, BC, IK, KL), any two quantities (CD, LM) that have the same ratio to the respective antecedents be added; these sums and the antecedents will still be proportionals (I say, if $AB : BC :: IK : KL$, and $AB : CD :: IK : LM$; then shall $AB : BD :: IK : KM$.)

For,

For, CD and LM
being proportional
to the antecedents
• Hyp. AB and IK; they
and the consequents
(BC, KL) will also be proportionals (by Corol. 2.
3. 4. of the precedent): whence (by composition) $BC : BD :: KL : KM$. And so again, (by the same Corol.) $AB : BD :: IK : KM$.

COROLLARY.

From this Theorem it will appear, that, if the ratios of the corresponding quantities of two ranks with respect to the two first, are the same in both ranks ($AB : BC :: IK : KL$, $AB : CD :: IK : LM$, &c.), then the ratio of all the quantities to the first, will also be the same in the one rank, as in the other. For, by adding DE and MN to the last consequents (BD, KM) there will be had $AB : BE :: IK : KN$ (and so on, to any number of quantities whatever.) Then (by composition) $AB : AE :: IK : IN$.

When the quantities in both ranks, are of the same kind, it will appear (by alternation) that the ratio of the two sums, and that of every two corresponding terms, will be the same.

The six Theorems here delivered, on proportions of magnitudes in general, comprehend all that is most useful in that subject:—What relates to the proportions of extended magnitudes; under different limitations, and figures, as far as regards right lines and surfaces, will be the subject of the remaining part of this book.

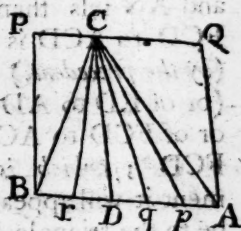
Note,

Note, whenever, in any demonstration, you meet with several proportional quantities, connected continually by the sign $::$ (like these, $A : B :: C : D :: E : F :: G : H$) the conclusion to be drawn, is always from the first and last of the two equal ratios.

THEOREM VII.

Triangles (ACD, BCD), and also parallelograms (ADCQ, BDCP), having the same altitude, are to one another in the same ratio as their bases (AD, BD).

Let the base AD be to the base BD in the ratio of any one number m (3) to any other number n (2), or which is the same^c, let AD contain m (3) such equal parts whereof BD contains the number n (2).



^c Ax. 3 4.

Then, the triangles ACp, pCq, BCr, &c. made by drawing lines from the points of division to the vertex C, being all equal among themselves^d; the triangle ACD will therefore be in proportion to the triangle BCD, as the number of equal parts in the former to the number of equal parts in the latter, or as the number of parts in AD to the number of parts in BD, that is, as AD to BD^e, whence, also, the parallelograms ADCQ, BDCP, being the doubles of their respective triangles^d, are likewise in the same ratio as their bases AD and BD^e.

^d Cor. 2. to 2. 2.

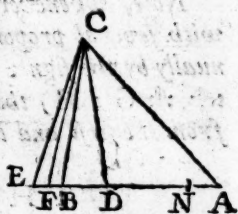
^e 4 4.

SCHOLIUM.

If the bases AD and BD are incommensurable to each other, the ratio of the triangles cannot be other than that of their bases.

For,

For, if possible, let the triangle BCD be to the triangle ACD, not as BD to AD, but as some other line ED, greater than BD, is to AD.

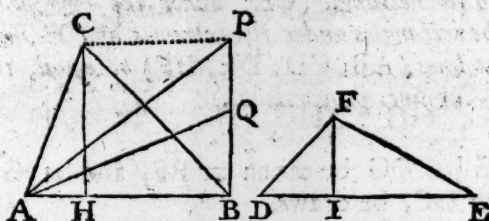


- Let AN be a part, or measure of AD, less than BE^s, and let DF be that multiple of AN which least exceeds DB; also let CE and CF be drawn. It is manifest that the point F falls between B and E, because (by Hyp.) BF is less than AN, and AN less than BE. Moreover, the ratio of FCD to ACD is the same as that of FD to AD (by the precedent.) But the ratio of BCD to ACD (or of ED to AD^f) is $\sqsubset$ the ratio of FD to AD^s, or of FCD to ACD; and consequently BCD $\sqsubset$ FCD^h; which is impossibleⁱ; by the same argument it will appear, that the triangle BCD cannot be to the triangle ACD, as a line, less than BD is to AD. Therefore BCD : ACD :: BD : AD.
- ^f Hyp.
^s Ax. 5.
 of 4.
^h Ax. 6.
 of 4.
ⁱ Ax. 2.
 of 1.

If this Scholium should appear difficult to the Learner, it may not be amiss to omit it intirely; since it is only put down for the sake of those who may be scrupulous about the business of incommensurables; to whom it may not be improper to observe, that nothing more is taken for granted herein, than what is effected by means of the first Lemma in the 8th book; which being demonstrated from axioms, and one single theorem in the first book, is referred to here, though not given till hereafter, for reasons already hinted at, in this note.

THEOREM VIII.

Triangles (ABC, DEF) standing upon equal bases (AB, DE) are to one another, in the same ratio as their altitudes CH, FI.)



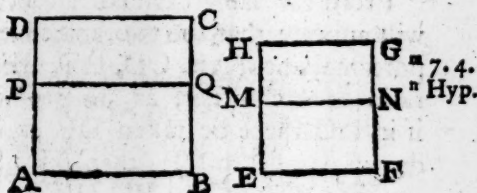
Let BP be perpendicular to AB; and equal to CH; in which let there be taken BQ = FI, and let AP and AQ be drawn.

The triangle ABP is = ABC^k, and ABQ = DEF^k; but ABP (ABC) : ABQ (DEF) :: BP (HC) : BQ (FI). ^{Cor. 2. to 2. 2. 7. 4.}

THEOREM IX.

If, parallel to the bases of any two parallelograms (AC, EG), two lines (PQ, MN) be drawn, so as to cut the sides proportionally (AP : AD :: EM : EH), then will those parallelograms and their corresponding parts (AQ, EN) be also proportionals.

For, AQ : AC
:: AP : AD^m ::
EM : EHⁿ :: EN
: EG^m; and there-
fore, by alternation,
AQ : EN :: AC
: EG^o.



^{7. 4. Hyp. 2. 4.}

THEO.

THEOREM X.

If four lines are proportional ($AB : CD :: DE : BF$), the rectangle (AF) under the two extremes will be equal to the rectangle (CE) under the two means. And, if the rectangles under the extremes and means of four given lines (AB, CD, DE, BF) be equal, then are those four lines proportional.

In DE let DG be taken $= BF$, and let GH, parallel to DC, be drawn.

1. Hyp. $AF : CG$
 P 7. 4. $:: AB : CD :: DE : BF$
 Hyp. $: BF : DG$
 Ax. 1. 4. $: CE : CG$; therefore, the consequents of the first and last of these equal ratios being the same quantity CG, the two antecedents AF and CE must be equal.
 Ax. 3. Hyp. $AB : CD :: AF : CG :: CE : CG :: DE : DG$ (BF.)
 Ax. 2. 4.



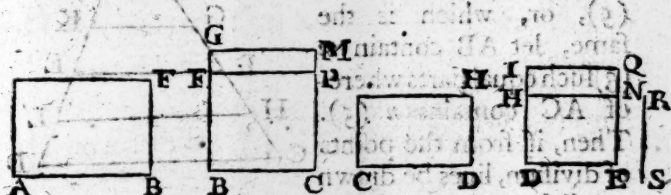
SCHOLIUM.

- From the same demonstration, and scheme, it will appear, that the two antecedents of four proportional lines (AB, CD, DE, BF) are in the same ratio to each other, as the two consequents: for, if in DC there be taken DP $= BF$, and PQ be drawn parallel to DE; then $AB : DE :: AF : PE :: CE : PE :: CD : PD$ (PF).
 7. 4.

THEOREM XI.

The rectangles under the corresponding lines, of two ranks of proportionals, are themselves proportionals. (I say, if $AB : BC :: CD : DE$, and $BF : BG :: DH : DI$, then will the rectang. $AF : \text{rectang. } BM :: CH : \text{rect. } DQ$).

For, in BC and DI (produced if necessary) let there be taken $BF = BF$, $DH = DH$, and let



FP be parallel to BC , and HN to DE : then $AF : BP :: AB : BC :: CD : DE :: CH : DN$, whence (alternately) $AF : CH :: PB : DN$, and so likewise is BM to DQ : whence (again by alternation) $AF : BM :: CH : DQ$.

COROLLARY I.

Hence, the squares of four proportional lines, are themselves proportional.

COROLLARY II.

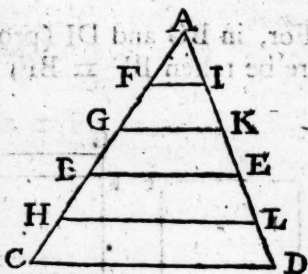
Hence also, the sides of four proportional squares (AB^2, BC^2, CD^2, DE^2) will be proportional. For, let the line RS be taken such, that, $AB : BC :: CD : RS$, then, since $AB : BC :: CD^2 : RS^2$ (by Corol. 1.) and $AB^2 : BC^2 :: CD^2 : DE^2$ (by sup-^b Ax. 4. 4. position). thence will ^b $RS^2 = DE^2$; therefore ^c $RS = DE$, and consequently $AB : BC :: CD : DE$. ^d Ax. 1. 4.

THEO.

THEOREM XII.

A line (BE) drawn parallel to one side (CD) of a triangle (ACD) divides the other two sides proportionally. (I say, $AB : AC :: AE : AD$, $AB : BC :: AE : ED$, and $AC : BC :: AD : ED$).

Let AB be to AC, as any one number m (3) is to any other number n (5), or, which is the same, let AB contain m (3) such equal parts where-
of AC contains n (5). Then, if from the points of division, lines be drawn



parallel to the side CD, they will also divide AE

* Cor. 1. and AD into the like number of equal parts^e,
to 27. 1. therefore AE is to AD, as the number of equal

parts in AE to the number of equal parts in AD, or as the number of equal parts in AB to the number of equal parts of AC, that is, as AB

† Ax. 8. to AC^f. In the same manner, AE is to ED, as the number of parts in AE to the number of parts in ED, or as the number of parts in AB to the number of parts in BC, that is, as AB to BC. Also, in the same manner, $AC : BC :: AD : ED$.

The same otherwise.

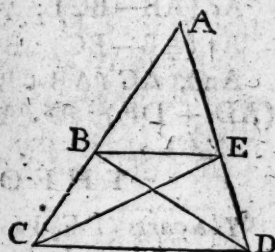
Draw CE and DB. Then will the triangles BEC

† Cor. 1. and EBD be equal to each other^f; whence, by
to 2. 2. adding BEA to both, AEC will be also = ABD^g.

‡ Ax. 4. 1.

But,

But, $AB : AC ::$ triang.
 $AEB : \text{triangle } AEC^b$
 $(ABD) :: AE : AD^b$; and
 $AB : BC ::$ triang. $AEB :$
 $\text{triangle } CEB^b (DEB) ::$
 $AE : ED.$



^b 7. 4.

COROLLARY.

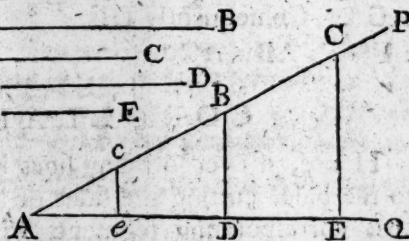
Hence a right line, which divides two sides of a triangle proportionally, is parallel to the remaining side: because AD is divided in the same ratio with AC, when BE is parallel to CD; but not else^f.

ⁱ Ax. 2. 1.
 and 5. 4.

SCHOLIUM.

From this last Theorem, whatever relates to the composition and division of ratios, when these respect the comparison of right-lines will appear exceedingly obvious.

For, let AB, A ————— B
 BC, AD, and B ————— C
 DE, be pro- A ————— D
 portionals: and D ————— E
 from any point A, let two indefinite right-
 lines AP, AQ



be drawn; in which take $AB = AB$, $BC = BC$, $AD = AD$, and $DE = DE$; also take $Bc = BC$, $De = DE$, and let BD, CE, and ec be drawn.

Since $AB : BC :: AD : DE^k$, thence is EC pa-^k Hyp.
 rallel to DB^l ; and so, Bc being $= BC$, and De^l Cor. of
 $= DE$, ec will also be parallel to DB^m . Therefore, ^{12. 4.}
^m Cor. 2. to 27. 1.

$AC (AB + BC) : AB :: AE (AD + DE) : AD$;
 $AC (AB + BC) : BC :: AE (AD + DE) : DE$;

G

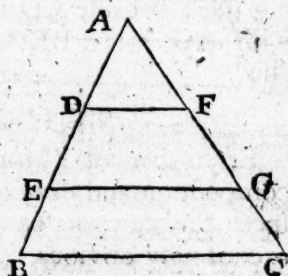
As

$Ac(AB-BC) : AB :: Ae(AD-DE) : AD$;
 $Ac(AB-BC) : BC :: Ae(AD-DE) : DE$;
 And, $AC(AB+BC) : Ac(AB-BC) :: AE$
 $(AD + DE) : Ae(AD-DE)$.

THEOREM XIII.

The parts (DE, FG) of the two sides of a triangle, intercepted by right-lines (DF, EG) drawn parallel to the base (BC), are in the same ratio with the wholes. (DE : FG :: AB : AC).

For, DF and EG
 being parallel to each
 other, thence will DE
 : AE :: FG : AG ° ;
 therefore, by (alterna-
 tion) DE : FG :: AE :
 AG °. In the same man-
 ner, AE : AG :: AB :
 AC °. Consequently DE
 : FG :: AB : AC °.



COROLLARY.

Hence, if ever so many lines be drawn parallel to the base, cutting the sides of a triangle, every two corresponding segments will have the same ratio °.

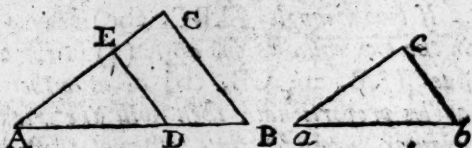
THEOREM XIV.

In triangles (ABC, abc) mutually equiangular the corresponding sides (AB, ab, AC, ac) containing the equal angles (A, a) are proportional.

In

In AB and AC (produced if necessary) take $AD = ab$, and $AE = ac$, and join D, E.

The triangles abc and ADE, having $ab = AD$, $ac = AE$,



and the angle $a = A$, have also the angle $ADE = abc = ABC$; whence DE will be parallel to BC; and therefore $AB : AD (ab) :: AC : AE (ac)$.
 Ax. 10.
 Hyp.
 Cor. to 8. 1.
 12. 4.

The same otherwise.

Because $AB \times AC = ab \times AC$, therefore is $AB : ab :: AC : ac$.
 24. 3.
 10. 4.

COROLLARY.

Hence equiangular triangles are similar to each other.
 Def. 14 of 4.

THEOREM XV.

If two triangles (ABC, abc) have one angle (BAC) in the one, equal to one angle (bac) in the other, and the sides (AB, ab, AC, ac) about those angles proportional; then are the triangles equiangular.

In AB and AC, take $AD = ab$, and $AE = ac$, and let DE be drawn.



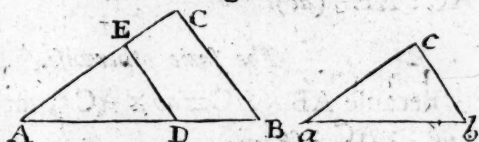
Since $AB : ab (AD) :: AC : ac (AE)$, therefore is DE parallel to BC; whence the angle $B = ADE$, and the angle $C = AED$.
 Hyp.
 Cor. to 12. 4.
 Cor. 1. to 7. 1.
 Ax. 10.
 1.

THEOREM XVI.

If two triangles (ABC , abc) have one angle (A) in the one, equal to one angle (a) in the other, and the sides (AB , ab , CB , cb) about either of the other angles proportional; then will the triangles be equiangular, provided these last angles (B , b) be, either, both less, or both greater, than right-angles.

In AB , let AD be taken $= ab$, and let DE be drawn parallel to BC , meeting AC in E .

Then will the triangles ABC , and ADE , be equiangu-



lar^f; therefore, $CB : ED :: AB : AD$ ^g :: $AB : ab$ ^h :: $CB : cb$ ⁱ; and consequently $ED = cb$ ^k; whence the triangles abc and ADE (having $ab = AD$, $cb = ED$, and $a = A$) will be equal in all respects^l, provided the angles abc and $ABC (= ADE)$ are either both less, or both greater than right-angles. Therefore, since the latter of these equal triangles (abc , ADE) is equiangular to ABC , the proposition is manifest.

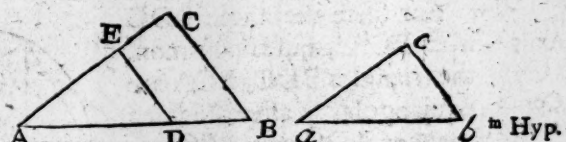
THEOREM XVII.

If two triangles (ABC , abc) have all their sides, respectively proportional ($AC : ac :: AB : ab :: CB : cb$) then are those triangles equiangular.

In AC and AB , take $AE = ac$, and $AD = ab$, and join E , D .

Since

Since AC
 $: AE(ac)::$
 $AB : AD$
 $(ab)^m$, the
 triangles

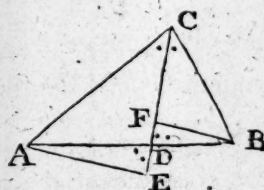


ABC, ADE are equiangularⁿ; hence $CB : ED ::$ ⁿ 15. 4.
 $AB : AD^o (ab) :: CB : cb^m$; and consequently^o 14. 4.
 $ED = cb^p$: therefore the triangles abc, ADE , ^p Ax. 4. 4.
 being mutually equilateral, they must also be mu-
 tually equiangularⁿ; and consequently abc , as wellⁿ 14. 1.
 as ADE , equiangular to ABC .

THEOREM XVIII.

A right-line (CD) bisecting any angle (ACB) of a triangle (ABC) divides the opposite side (AB) into two segments (AD, BD) having the same ratio with the sides (AC, CB) containing that angle.

Let AE and BF be perpendicular to CDE . Then the triangles ACE, CBF , and ADE, BDE being, respectively, equiangularⁿ, it will be $AD : BD :: AE : BF^p :: AC : BC$.

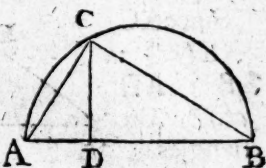


^p Hyp.
 and 3. 1.
^o 14. 4.
 and 2. 4.

THEOREM XIX.

A perpendicular (CD) let fall from the right-angle (C) upon the hypotenuse (AB) of a right-angled triangle (ABC) will be a mean proportional between the two segments (AD, BD) of the hypotenuse: and each of the sides containing the right-angle, will be a mean proportional between its adjacent segment, and the whole hypotenuse.

- For since the angle BDC
 Ax. 7. is = BCA', and B common,
 the triangles BDC, BCA are
 Cor. 1. equiangular": after the same
 10.10.1. manner ADC and ABC ap-
 pear to be equiangular.



Therefore, by Theor. XIV,
 $BD : CD :: CD : AD$
 $AB : BC :: BC : BD$
 $AB : AC :: AC : AD.$

COROLLARY.

- Because the angle in a semi-circle is a right-
 13. 3. angle", it follows, that, if from any point C, in
 the periphery of a semi-circle ACB, a perpendi-
 cular CD be let fall upon the diameter AB, and
 from the same point C, to the extremities of that
 diameter, two chords CA, CB be drawn; the square
 of that perpendicular will be equal to a rectangle
 under the two segments of the diameter; and
 the square of each chord, equal to a rectangle
 under the whole diameter and its adjacent segment:
 for, because of the above proportions, we have
 $CD^2 = BD \times AD$, $BC^2 = AB \times BD$, and $AC^2 =$
 19. 4. $AB \times AD.$

THEOREM XX.

If, in similar triangles (ABC, EFG) from any two equal angles (ACB, EGF) to the opposite sides, two right-lines (CD, GH) be drawn, making equal angles with the homologous sides (CB, GF); those right-lines will have the same ratio as the sides (AB, EF) on which they fall, and will also divide those sides proportionally.



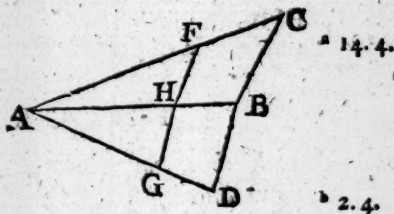
For, the triangles ADC, EHG, and BDC, FHG (as well as the wholes ABC, EFG) being equiangular.

thence is $AB : EF :: AC : EG :: CD : GH$; ^{Hyp.} $AD : EH :: DC : HG :: BD : FH$. ^{14. 4. and Ax. 7.}

THEOREM XXI.

If in two triangles (ABC, ABD) having one side (AB) common to both, from any point H in that side, two lines (HF, HG) respectively parallel to two contiguous sides (BC, BD) be drawn, to terminate in the two remaining sides (AC, AD); those lines (HF, HG) will have the same ratio as the sides (BC, BD) to which they are parallel.

For, $AB : AH :: BC : HF^a$, and $AB : AH :: BD : HG^a$; therefore, by equality, $BC : HF :: BD : HG$; whence, alternately, $BC : BD :: HF, HG^b$.



COROLLARY.

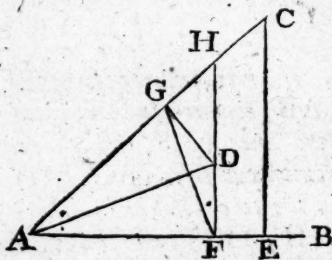
Hence, if $BC = BD$, then also will $HF = HG$.

THEOREM XXII.

If, at any two points (F, G) in two lines (AB, AC) meeting each other, two perpendiculars (FD, GD) be erected, so as to meet each other; the distance (AD) of their concurrence from that of the proposed lines, will be to the distance (FG) of the two points themselves, in the ratio of one of the said lines (AC) to a perpendicular (CE) falling from the extreme thereof upon the other (AB).

Let FD be produced to meet AC in H.

Since the angles AFD and AGD are right-ones^c, the circumference of a circle will pass thro' all the four points A, F, D, G^d; and so the angles GFD, GAD, standing on the same subtense GD, will be equal^e; and consequently the triangles AHD, FHG equiangular^f: therefore $AD : FG :: AH : HF$ ^g :: $AC : CE$.



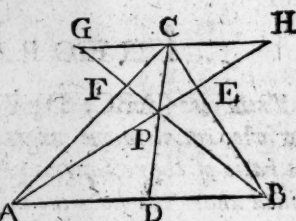
THEOREM XXIII.

If thro' any point (P) in a triangle (ABC) three right lines (AE, BF, CD) be drawn, from the angular points to cut the opposite sides, the segments (AD, BD) of any one side (AB) will be to each other, as the rectangles (AF × CE, BE × CF) under the segments of the other sides taken alternately.

Let

Let GCH be parallel to AB, and let AE and BF be produced to meet it in H and G.

It is manifest, that the triangles EBA, FCG; EAB, EHC; APB, A



GPH, are equiangular^h.

Therefore $AF : CF :: AB : CG^1$,
 $CE : BE :: CH : AB^1$.

^h 7. and 9.
of 1.

ⁱ 14. 4.

Whence $AF \times CE : CF \times BE :: CH \times AB : CG \times AB^k :: CH : CG^1$; but $CH : CG :: AD : BD^m$ ^{k 11. 4.}
therefore, by equality, $AF \times CE : CF \times BE :: AD : BD.$ ^{1 7. 4. m 20. 4.}

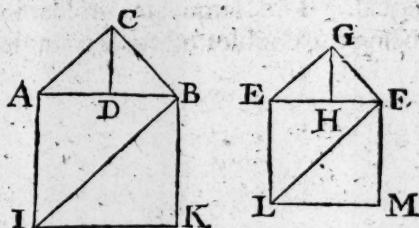
COROLLARY.

Hence, if $AD = BD$, then also will $AF \times CE = CF \times BE$, and therefore $AF : CF :: BE : CE^n$. ^{n 10. 4.}

THEOREM XXIV.

Equiangular triangles (ABC, EFG) are in proportion to one another, as the squares (AK, EM) of their homologous sides.

Upon AB and EF let fall the perpendiculars CD and GH, and let the diagonals BI, FL be drawn.

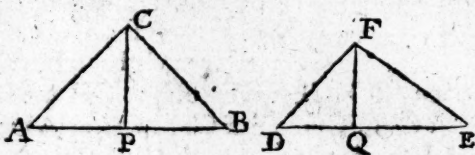


Because $ABC : ABI :: CD : AI$ (AB)^o :: $GH^p : (EF)$ (EL)^o :: $EFG : EFL^o$; therefore, alternately, ^{p 8. 4. o 20. 4. q 2. 4. r Cor. 2. to 2. 2.}
 $ABC : EFG :: ABI : EFL^q :: AK : EM^r$.

THEO-

THEOREM XXV.

Triangles (ABC, DEF) having one angle (A) in the one, equal to one angle (D) in the other, are in the ratio of the rectangles ($AC \times AB$, $DF \times DE$) contained under the sides including the equal angles.



Upon AB and DE, let fall the perpendiculars CP and FQ. Then $AC \times AB : CP \times AB :: AC : CP :: DF : FQ :: DF \times DE : FQ \times DE$; whence, alternately, $AC \times AB : DF \times DE :: CP \times AB : FQ \times DE :: \text{triangle } ABC : \text{triangle } DEF$.

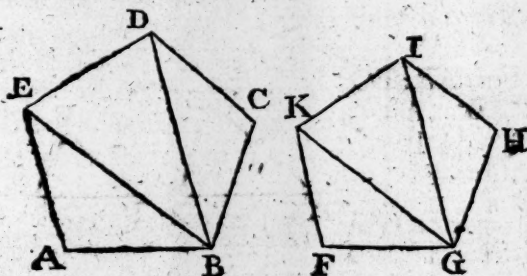
COROLLARY.

Hence, if the rectangles of the sides containing the equal angles, be equal, or the sides themselves reciprocally proportional, the triangles will be equal. The same also holds in parallelograms, being the doubles of such triangles.

THEO.

THEOREM XXVI.

All similar right-lined figures (ABCDE,FGHIK) are in proportion to one another as the squares of their homologous sides (AB, FG).



Draw the right-lines BE, BD, GK, GI.

Because $A = F$, and $AB : AE :: FG : FK$ ^a, the ^a Def. 14. triangles BAE, GFK are equiangular^y; therefore, ^{of 4.} if from $AED = FKI$ ^x, there be taken $AEB =$ ^y 15. 4. FKG , the remainders BED, GKI will also be equal^z. ^z Ax. 5. Wherefore, since $ED : KI (:: EA : KF$ ^x)^z : EB ^{of 1.} : KG ^w, the triangles EBD, KGI are likewise equi-^w 14. 4. angular^y. In the same manner it will appear, that DBC, IGH are also equiangular.

Therefore, because $ABE : GFK (:: BE^2 : GK^2)$ ^a 24. 4. : $BED : GKI (:: BD^2 : GI^2) :: BDC : GIH$, it is evident, that the sum of all the antecedents (ABCDE) is to the sum of all the consequents (FGHIK) as the first antecedent ABE is to the first consequent GFK^b, or as AB^2 to FG^2 ; which^b Cor. to 6. 4. was to be demonstrated.

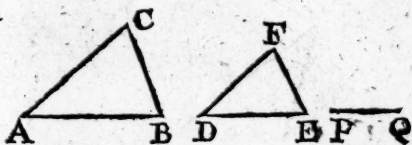
THEO-

THEOREM XXVII.

If three right-lines (AB, DE, PQ) are proportional, the right-lined figure (ABC) upon the first, will be in proportion to the similar, and similarly described, figure (DEF) on the second, as the first line (AB) to the third (PQ)

For, AB : PQ
 :: AB² : AB ×
 PQ^c (^d DE²) ::
 ABC : DEF^e.

^c 7. 4.
^d Hyp.
 and 10.
 of 4.
^e 26. 4.



COROLLARY.

Hence, similar right-lined figures, are in the duplicate ratio of their homologous sides^f.

^f Def. 7.
 of 4.

THEOREM XXVIII.

If four right-lines (AB, CD, EF, GH) be proportional, the right-lined figures described upon them, being like, and in like sort situate, shall also be proportional (ABI : CDK :: EM : GO).



For, ABI : CDK (:: ^g AB² : CD² :: EF² : GH² ^h)
 :: EM : GO^g.

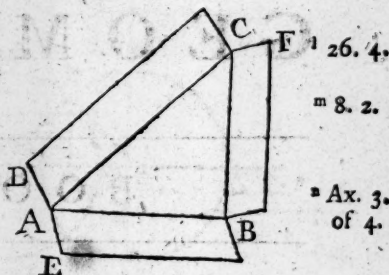
^g 26. 4.
^h Cor. 1.
 to 11. 4.

THEO.

THEOREM XXIX.

If upon the three sides of a right-angled triangle (ABC) as many right-lined figures (CD, BE, BF) like, and alike situate, be described, that (CD) upon the hypotenuse (AC) will be equal to both the other two (BE, BF) taken together.

For, $BE : BF :: AB^2 : BC^2$; therefore (by composition) $BE + BF : BE :: AB^2 + BC^2 (= AC^2) : AB^2 :: CD : BE$; and consequently $BE + BF = CD$.



The End of the FOURTH BOOK.

E L E M E N T S

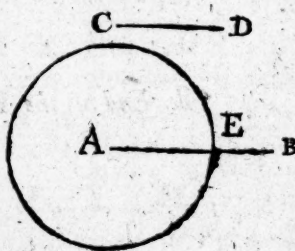
O F

G E O M E T R Y.

B O O K V.

P R O B L E M I.

FROM the greater
(AB) of two unequal
lines (AB, CD) to
cut off, or take away a
part (AE) equal to the
lesser (CD).



From A as a center,
with a radius equal to
CD, let the circumference

* Post. 3. of a circle be described *

* Ax. 2. cutting * AB in E; and the thing is done.

P R O B L E M I I.

*At a given point (A) to make a line (AB) equal
to a given line (CD).*

Draw

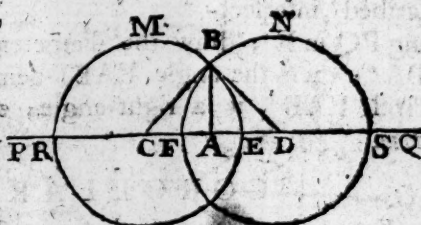
$\begin{array}{c} C \text{ --- } D \\ A \text{ --- } F \\ \quad B \end{array}$
 Draw the indefinite line AF^c; from which take away AB = CD^d; and the thing is done.

Post. 1.
 and 2.
 1. 5.

PROBLEM III.

At a given point (A) in an infinite right-line (PQ) to erect a perpendicular.

In the line proposed, take two equal distances AC, and AD^e; and from the centers C^f and D^g, with any equal radii great-



1. 5.
 Post. 3.
 Post. 1.

er than AC (or AD), let two circles EMR and FNS be described; which will cut each other: and, if from the point B of their intersection, you draw BA, the thing is done.

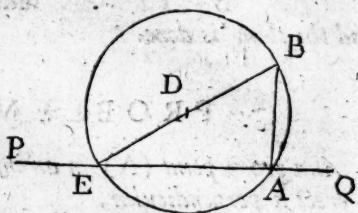
For let the points R, E, and F, S, be those wherein the infinite line PQ intersects the circumferences of the two circles EMR and FNS^h: then AF being $\sqsupset$ FD^h (or CRⁱ) $\sqsupset$ AR^h; and AS $\sqsubset$ DS^h (or CEⁱ) $\sqsubset$ AE^h, the point F falls within, and the point S without the circle EMR; and so the two circles cut each other^l. If therefore, from the point of intersection, BC and BD be drawn; then the triangle CBD being isosceles^m, the angles BCD, BDC at the base thereof, will be equalⁿ; and so, CA being = AD^k, and CB = BD^m, the angle CAB is also = DAB^p.

Def. 11.
 3.
 Constr.
 and Def.
 33. of 1.
 Obs. on
 Ax. 10.
 Constr.
 Ax. 10.

Otherwise.

Otherwise.

From any point D above the line PQ, as a center, thro' the given point A, let the circumference of a circle be described, intersecting PQ in E; draw the diameter EDB, and also BA; then the angle EAB, being in the semicircle EAB, is a right-angle; *which was to be done.*



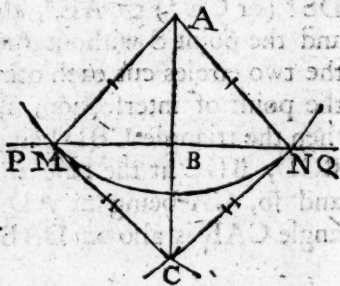
COROLLARY.

From the former of these constructions it appears, that, if from any two points, with two equal radii, greater, each, than half the distance of those points, two circles be described; those circles will cut each other.

PROBLEM IV.

From a given point (A) upon an infinite right-line (PQ) to let fall a perpendicular (AB).

From the given point A, as a center, let an arch of a circle be described, so as to pass below PQ and intersect it in M and N; from which points, with any equal radii, greater than half MN, let two other arches be also described, and from the point of



their

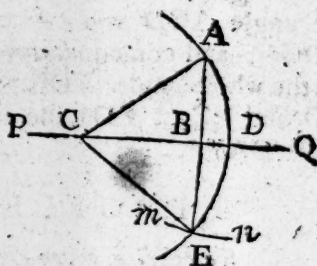
their intersection C, let the right-line CBA be drawn; which will be perpendicular to PQ.

For, let AM, AN, CM, and CN be drawn; then AM being = AN^a, and MC = NC^a, the angle AMB is = ANB^a, and CMB = CNB^a; and consequently AMC = ANC^a: whence, (as AM = AN, and MC = NC) the triangles AMC, ANC are equal in all respects^a; and so, the angle MAB being = NAB, the angle MBA is likewise = NBA^a.

^a Def. 33.
of 1.
Hyp.
and Def.
33. 1.
^a Obj. on
Ax. 10.
^a Ax. 4.
^a Ax. 10.

The same otherwise.

From any point C in the line PQ, as a center, let the circumference of a circle be described thro' the given point A^a, intersecting PQ in D^b; and from the center D, with a radius equal



^a Post. 3.
^b Constr.

to the distance of the points A and D, let another circle *mEn* be also described, cutting the former ADE in E; then draw ABE for the perpendicular required.

For, conceiving right-lines to be drawn from C and D, to A and E, the triangles ACE, ADE will be both of them isosceles^b; and so the demonstration is the same with that of the preceding method.

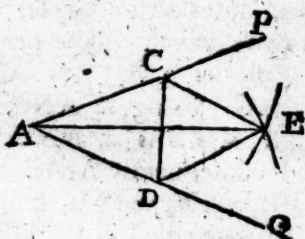
PROBLEM V.

To bisect, or divide into two equal parts, any given right-lined angle (PAQ).

H

I

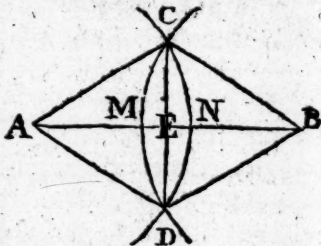
- In the lines containing the given angle, take $AC = AD^e$; and upon the centers C and D, with any equal radii, let two circles be described^d, so as to intersect each other; and from the point of intersection E draw EA, *and the thing is done.*
- ^e Post. 1. For, let CD, CE, and DE be drawn^e; then, the triangles ACD and ECD being both isosceles^f, and Def. the angle ACD will be $= ADC$, and $ECD = EDC^g$; and consequently the whole angle ACE = the whole angle ADE^h: whence (AC being $= AD$, and $EC = ED$), the angle CAE is also $=$ the angle DAEⁱ.



PROBLEM VI.

To bisect a given right-line (AB).

- From the extremes A, B, of the given line, with equal radii, describe two circles, so as to cut each otherⁱ; and between the two points of intersection draw CD, cutting AB in E; *and the thing is done.*
- ⁱ Post. 3. For, if AC, AD, BC and BD be drawn, the triangles ACB, ADB being isosceles^k, thence is 33. of 1. the angle CAB $= CBA^l$, and DAB $= DBA^l$; and consequently CAD $= CBD^m$: whence the triangles ACD and BCD are equal in all respectsⁿ; and to the angle ACE being $= BCE$, AC $= BC$, and CE common, thence is AE also $= BE^n$.



COROL.

COROLLARY.

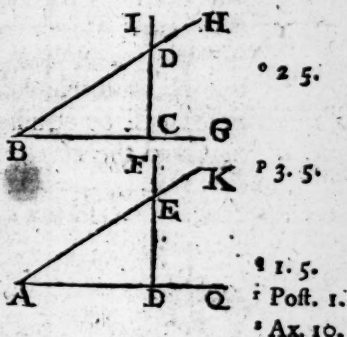
Hence, it is manifest, that CD not only bisects AB, but is also perpendicular to itⁿ.

ⁿ Ax. 10.
and Def.
8. of 1.

PROBLEM VII.

From a given point (A) in a given right-line (AQ) to draw a line (AK) which shall make with the former an angle, equal to an angle given (HBG).

In BG and AQ take two equal distances BC, AD^o; and at C and D erect the two perpendiculars CI and DF to BG and AQ^p; and in DF take DE equal to the part CD of the former, intercepted by the lines containing the given angle HBG^q; then thro' E draw AEK^r, and the thing is done^s.



SCHOLIUM.

Having, in the seven preceding problems, effected and demonstrated, by means of the axioms only, whatever was assumed in the fourth postulate, as barely possible; we are now authorized, by the most rigid laws of geometrical reasoning, to make use of any theorem or conclusion, whatsoever, derived in the preceding books, in virtue of those assumptions, by which the process and result can be rendered the most obvious and eligible.—Accordingly, by having recourse here to Theorem XIV. of the first book, we shall be able to arrive at a construction of the last problem, better adapted to practice than that above laid down.

H 2

From

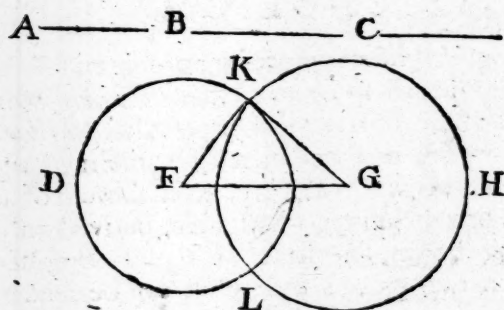
From the centers A and B, at any equal distances AD, BF, let two arcs of circles, FC, DR, be described, intersecting the given lines in D, E, and C; also from D, with a radius equal to the distance of the points F, C, let another circular arch pEr be described, cutting the former DR, in E; then draw AEK, and the thing is done.



For, conceiving right-lines to be drawn from F to C, and from D to E, the triangles BFC and ADE will be equilateral to each other, by construction, and therefore equiangular also.

PROBLEM VIII.

To describe a triangle, whose three sides shall be equal to three given lines (A, B, C); provided any two of them, taken together, be greater than the third.

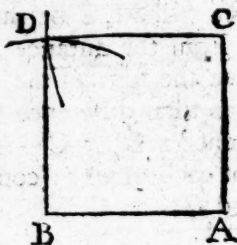


Make $FG = B$, and from the centers F and G, with the intervals, or distances A and C, let two

PROBLEM X.

Upon a given line (AB) to describe a square (ABCD).

- Make AC perpendicular,
 • 3. 5. and equal to AB^o; and from
 1. 5. the centers B and C, let two
 circles, with the radius AB
 • Post. 3. or AC, be described^p; inter-
 • 9. 3. secting each other^q in D;
 from which point draw DB
 and DC, and the thing is
 done.



- For, all the four sides being equal, by construc-
 tion, the figure is a parallelogram^r; and therefore
 the angle A being a right-angle^s, the other three
 will be all right-angles^t, and ACDB a square^u.
 • 25. 1.
 • Constr.
 • Cor. to
 24. 1.
 • Def. 26.

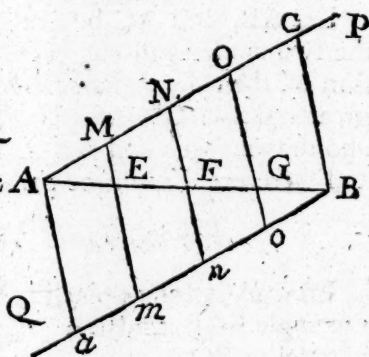
SCHOLIUM.

By the same method a rectangle may be de-
 scribed, the sides thereof being given.

PROBLEM XI.

To divide a given line (AB) into any proposed num-
 ber of equal parts.

- From the extremes
 of the given line
 AB, draw two inde-
 finite lines AP, BQ
 parallel to each o-
 ther^v; in each of
 which lines let there
 be taken as many
 equal distances AM,
 MN, NO, OC; Bo,
 on, nm, ma, (of any



length

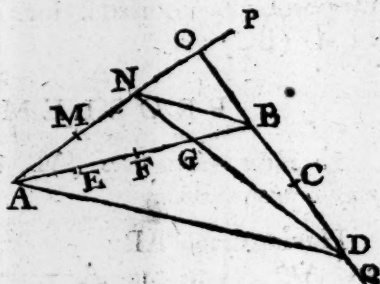
length at pleasure) as you would have AB divided into^v; then draw Mm, Nn, Oo, intersecting AB^v 1. 5. in E, F, G, and the thing is done.

For, MN and mn being equal and parallel^x, ^x Confr. FN will be parallel to EM^y; and in the same^y 26. 1. manner will GO be parallel to FN: therefore, AM, MN, NO, &c. being all equal^x, AE, EF, FG, will likewise be equal^z.

^z Cor. 1.
to 27. 1.

The same otherwisa.

In any right-line AP, drawn from A, take as many equal distances (AM, MN, NO) wanting one, as you would have AB divided into; then, having drawn the indefinite line OBQ, in it take an equal number of parts or distances OB, BC, CD, each of the length of OB, and let DN be drawn, cutting AB in G; make GF, FE, each equal to BG, and the thing is done.



For, if AD and BN be drawn, they will be parallel^a, (because $OA : ON :: OD : OB^b$); and so, the triangles BNG, ADG, being equiangular^c, it will be $BG : AG :: BN : AD^d :: ON : OA^d$. Therefore BG is the same part of AG, as ON is of OA.

^a Cor. to 12. 4.
^b Confr. 3. and 7. 1.
^d 14. 4.

PROBLEM XII.

To two given lines (AB, BC) to find a third proportional.

H 4

From

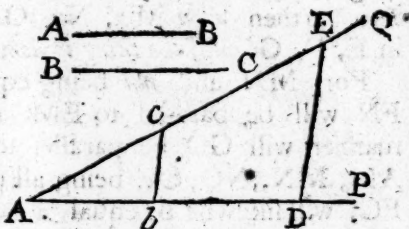
From any point
A, draw two in-
definite lines AP,
AQ, in which
take $Ab = AB$,
 $Ac = BC$, and
 $bD = BC^e$; draw

^e 1. 5.

^f 9. 5.

^h 12. 4.

bc , and parallel
to bc , draw DE^g ,
cutting AQ in E ; then cE will be the third
proportional required: for Ab (AB): Ac (BC)
 $:: bD$ (BC): cE^h .



PROBLEM XIII.

To three given lines (AB , AC , BD) to find a
fourth proportional.

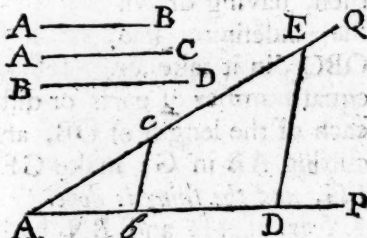
Having drawn AP
and AQ, as in the
preceding problem,
take therein $Ab =$
 AB , $Ac = AC$, and
 $bD = BD^i$; draw
 bc , and parallel to
it, draw DE^j , inter-
secting AQ in E ;
then is cE the fourth proportional required.

ⁱ 1. 5.

^j 9. 5.

^m 12. 4.

For, Ab (AB): Ac (AC): $:: bD$ (BD): cE^m .

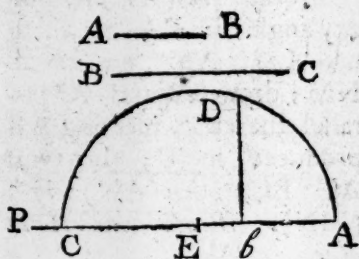


PROBLEM XIV.

Between two given lines (AB , BC) to find a mean
proportional.

In

In the indefinite line AP, take $Ab = AB$, and $bC = BC^n$; bisect AC in E°, and from the center E, at the distance of EA, or EC, let a semicircle ADC be described^p; erect bD , perpendicular to AC^q, cutting the circumference^q in D; then will bD be the mean-proportional required.



^p 1. 5.
^q 6. 5.

^p Post. 3.

^q 3. 5.

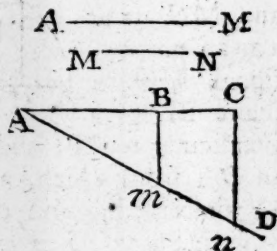
For, $Ab(AB) : bD :: bD : bC (BC)^r$.

^r 19. 4.
and it's
Corol.

PROBLEM XV.

To divide a given line (AC) into two parts (AB, BC) having the same proportion as two given lines (AM, AN).

From A draw AD, making any angle with AB; in which take $Am = AM$, and $mn = MN^s$; draw nC , and mB parallel thereto^t, meeting AC in B. Then will $AB : BC :: Am(AM) : mn^n(MN)$. Which was to be done.



^s 2. 5.

^t 9. 5.

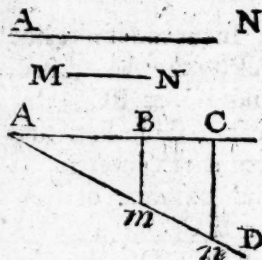
^u 12. 4.

PROBLEM XVI.

To add a line (BC) to a line given (AB), so that the whole compounded line (AC) shall be in proportion to the part added, as one given line (AN) is to another (MN).

From

- From A draw AD, making any angle with BA; in which
 2. 5. take $An = AN^w$, and $nm =$
 9. 5. NM ; draw mB' and nC pa-
 12. 4. rallel thereto^x, meeting AB
 produced, in C; then will
 $AC : BC :: An (AN) : mn^y$
 (MN). Which was to be
 done.



PROBLEM XVII.

To divide a given line (AB) into two such parts (AC, BC) that the rectangle contained under them, shall be equal to the rectangle under two given lines (PM, MN); provided that the given rectangle is not greater than the square of half the line (AB) to be divided.

- Between PM and MN take a mean-proportional MQ^z ; make BD perpendicular to AB, and equal to MQ ; bisect AB in O^z , from which, as a center, let a semi-circle be described; and draw DE parallel to BA^b, which (because BD is less than the radius^c) will meet the circle in some point E; from which, upon AB let fall the perpendicular EC: so shall
 14. 5. $AC \times BC = EC^2 = DB^2 (=QM^2) = PM \times$
 6. 5. MN^c . Which was to be done.



- Cor. to 19. 4. Hyp. and 10. 4.

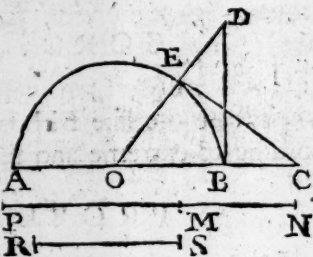
PROBLEM XVIII.

To a given line (AB) to add another line (BC), such that the rectangle under the whole compounded line (AC) and the part added, shall be equal to a rectangle under two given lines (PM, MN).

4

Between

Between PM and MN
take a mean-proportion-
al RS^f ; make BD
perpendicular to AB^g,
and equal to RS^h ; bisect
AB in Oⁱ, draw OD,
and take $OC = OD^k$:
so shall $AC \times BC = PM$
 $\times MN$, as was to be done.



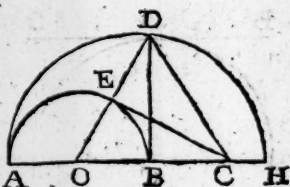
^f 14. 5.
^g 3. 5.
^h 2. 5.
ⁱ 6. 5.

For, if thro' A and B, from the center O, the
circumference of a circle be described, cutting
DO in E, and E, C be joined; then, the triangles
OCE, ODB (having $OE = OB$, $OC = OD^k$, ^k Confr.
and the angle EOB common) will be equal in all
respects^l; and so, EC being a ^m tangent to the ^l Ax. 10.
circle in E, we have ⁿ $AC \times BC = CE^2 =$ ^o BD^2 ^m 6. 3.
 $= RS^2 = PM \times MN^p$, ⁿ Cor. to
^{22.} 3.
^o 1. 2.
^p 10. 4.

PROBLEM XIX.

To divide a given line (BH) into two such parts,
that the square of the one (BC) shall be equal to the
rectangle under the other (CH) and a second given
line (AB)

Taking BA in the same
strait line with BH, between
them let a mean-propor-
tional BD be found^q; bi-
sect AB in O^r; Draw OD,
and make $OC = OD$; so
shall $BC^2 = CH \times AB$,
as was to be done.



^q 14. 5.
^r 6. 5.

For, by the demonstration of the precedent,
 $AC \times BC (= CE^2 = BD^2) = AB \times BH$; from each
of which taking away $AB \times BC$, there remains
 $BC^2 = AB \times CH^s$.

^s 5. 2. and
Ax. 5. 1.

COROL-

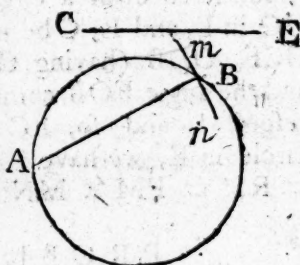
COROLLARY.

If $AB=BH$, then will $BC^2 = BH \times CH$; in which case the line BH is said to be divided according to extreme and mean-proportion.

PROBLEM XX.

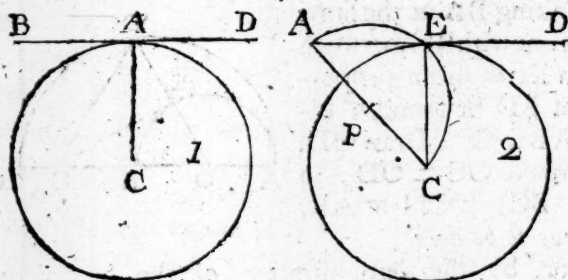
In a given circle, to apply, or inscribe a line (AB) equal to a given line (CE), less than the diameter of the circle.

- From any point A in the circumference, with the radius CE , let a circular arch mn be described, cutting the given circle in B^u ; then draw AB , and the thing is done v .
- * Post. 3.
 * 9. 3.
 * Def. 33. of 1.



PROBLEM XXI.

To draw a tangent to a given circle (C) thro' a given point (A).



- CASE I. If the given point be in the circumference; then, to the center C , draw AC , and perpendicular to AC draw BAD ; which will touch the circle at A .
- * 3. 5.
 * 6. 3.

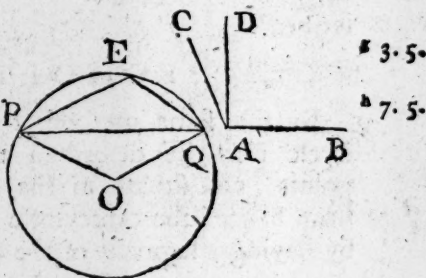
CASE II.

CASE II. If the point A be without the circumference; then draw AC, which bisect in P^b; and ^b 6. 5. from the center P, at the distance of AP, or CP, let a semi-circle AEC be described^c; cutting the ^c Post. 3. given circle in E^d; then draw AED, which will ^d 9. 3. be the tangent required^e; because (CE being ^e 6. 3. drawn) AEC is a right-angle^f. ^f 13. 3.

PROBLEM XXII.

Upon a given line (PQ) to describe a segment of a circle (PEQ) to contain an angle (E) equal to a given angle (BAC).

Make AD perpendicular to AB^s; also make PQQ, and QPO, each, equal to DAC^h P (the difference between the given angle and a right one); then upon the point of intersection O, as a center, at the distance of OP (or OQ), *and the thing is done.*



For the angle $E = \text{right-angle} + QPO^1 = 16.3$.
 $DAB + DAC^k = BAC$. k Contr.

SCHOLIUM.

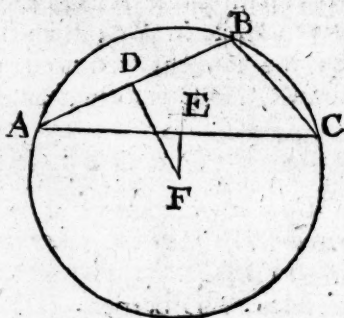
In the same manner the problem may be constructed, when the given angle is acute; only the lines PO, QQ must then be drawn on the other side of (PQ) as is manifest from the 16th theorem of the 3d book.

PRO-

PROBLEM XXIII.

About a given triangle (ABC) to describe a circle.

Let any two sides, AB and AC, be bisected by two perpendiculars DF and EF¹; which will intersect each other in the center (F) of the required circle¹; from whence the circle may be described.



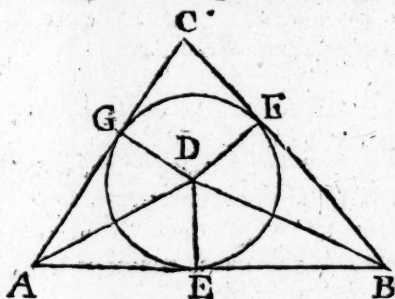
SCHOLIUM.

By the same method, the circumference of a circle may be described thro' any three given points, not situate in the same right-line: also from hence, the center of a circle may be found, by having a segment of the circle given.

PROBLEM XXIV.

To inscribe a circle in a given triangle (ABC).

Bisect any two of the angles, A and B, by the lines AD and BD^m, meeting each other in D; make DE perpendicular to AB^o; then, if from the center D, at the distance of DE, a circle be described, it will touch all the sides of the triangle.



For,

For, let DG and DF be perpendicular to AC and BC ; then the triangles ADE , ADG , having ^p 4. 5. two angles equal, each to each (by construction) and AD common, will not only be equiangular ^q, ^q Cor. 1. but also have $DE = DG$. By the same argument ^{to} 10. 1. $DE = DF$; therefore the circumference of the ^{15. 1.} circle also passes through G and F ; but it touches ^{Def. 33.} the sides of the triangle in those points ^{of} 1. G and F are right-angles ^{6. 3.}. ^{Constr.}

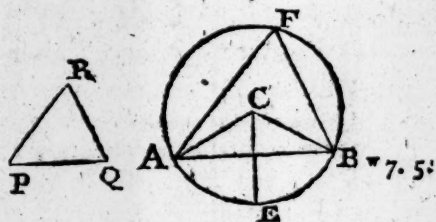
PROBLEM XXV.

In a given circle (AFB) to describe a triangle, equiangular to a given triangle (PQR).

From the center C , draw the radii CA , CE , CB , making the angles ACE and BCE equal, each to the angle R ; join A , B , and make the angle $ABF = Q$, and from the point F , where BF cuts the circle, draw FA ; so shall AFB be the triangle required.

For, $ABF = Q$, $F (= ACE) = R$; and ^{Constr.} consequently $BAF = P$.

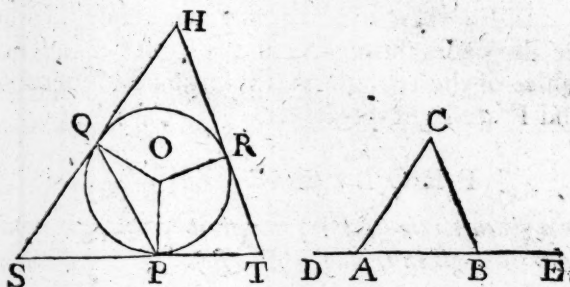
^{10. 3.}
; and
^{Constr.}
^{Cor. 1.}
^{to 10. 1.}



PRO-

PROBLEM XXVI.

About a given circle (O), to describe a triangle, equiangular to a given triangle (ABC).



- Produce out the side AB both ways; and draw the radii OP, OR, OQ, so as to make the angle $POR = EBC$, and $POQ = DAC$ ^a; then draw three right-lines to touch the circle in the points P, Q and R^b, and the thing is done.

For, if PQ be drawn, the angles SQP and SPQ, will be less than the two right-angles SQO and SPO^c; and so PS and QS, not being parallels^d, they will meet each other^d; therefore, as the like may be inferred with regard to PT and RT, &c. it is manifest that the three tangents form a triangle STH. Now, $POR + T$ being $=$ two right-angles^e $= ABC + EBC$ ^f, and $POR = EBC$ ^g; thence will $T = ABC$: and, by the same argument, $S = BAC$; whence also $H = C$.

^a Ax. 2. 1.
^b and 6. 3.
^c Cor. 2.
^d to 7. 1.
^e and
^f Cor. to
^g 5. 1.
^h Cor. 2.
ⁱ to 11. 1.
^j 1. 1.
^k Constr.

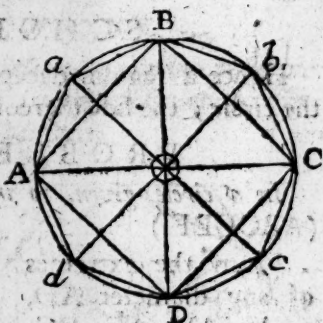
PROBLEM XXVII.

In a circle given (ABCD) to inscribe a square.

- Draw two diameters AC and BD perpendicular to each other^b; then draw AB, BC, CD and DA; so shall ABCD be a square inscribed in the circle.

For,

For, the angles AOB, BOC, DOC and DOA (as well as the sides OA, OB, OC, OD, containing them) being equal¹, the opposite sides AB, BC, CD, DA will likewise be equal²; and the angles ABC, BCD, CDA, DAB, are all of them right-angles¹, and therefore are equal.



¹ Ax. 7. 1. and Def. 33. 1.

² Ax. 10. of 1.

¹ 13. 3.

SCHOLIUM.

If two other diameters *ac*, *bd* be drawn (by *prob. 5.*) to bisect the angles AOB, BOC, a regular octagon *AaBbCcDd* may be inscribed in the circle. And if all the angles at the center *O*, be again bisected, a regular polygon of sixteen sides, may in like manner be determined; and so on, at pleasure.

PROBLEM XXVIII.

In a circle given (*ABGE*) to inscribe a regular pentagon.

At the center *O*, upon the diameter *FG*, erect the perpendicular *OB*^a, meeting the circumference in *B*; divide *OG* in *H* (by *prob. 19.*) so that $OH^2 = GH \times OG$; that is, take $OR = \frac{1}{2}OF$, and $RH = \text{dist. } RB$: Then draw *BH*; which will be equal to the side of the pentagon^a; from whence the figure itself may be described:



^m 3. 5.

^a 28. 3. and 8. 2.

I

SCHO-

SCHOLIUM.

Hence a regular decagon may be inscribed in the circle; the side thereof being $\approx OH$.

PROBLEM XXIX.

In a circle given, to inscribe a regular hexagon (ABCDEF)

From the extremes of any diameter AD, apply AB, AF, DC, and DE equal, each, to the radius AO^r; then join B, C, and E, F; and the thing is done.



For, if the radii OB, OC, OE, OF be drawn; the triangles AOB and DOC, being equilateral^r, will also have the angle OAB $\approx$ DOC^r; whence AB is parallel (as well as equal) to OC^r; and consequently BC and AO are likewise equal, and parallel^r: Therefore, seeing the triangles AOB, BOC, COD, &c. are equilateral, and alike in all respects; not only the sides, but also the angles ABC, BCD, &c. of the hexagon, will be equal among themselves^r.

COROLLARY.

Hence it appears, that the side of a regular hexagon, inscribed in a circle, is equal to the semi-diameter, or radius.

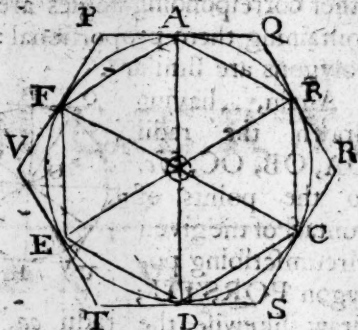
SCHOLIUM.

Besides the figures constructed in the preceding problems, and those arising from thence by continual bisections, or taking the differences, no other regular polygon can be described, from any known method, purely geometrical, by means of right-lines and circles only.

PROBLEM XXX.

About a given circle to describe a regular polygon, of the same number of sides with a regular polygon (ABCDEF) inscribed in the circle.

From the center O, to the angles of the inscribed polygon, draw OA, OB, OC, &c. and perpendicular thereto draw PAQ, QBR, RCS', &c. intersecting in P, Q, R, S, T, V; so shall PQRSTV be the polygon that was to be described.



3. 5.
Proof of
26. 5.

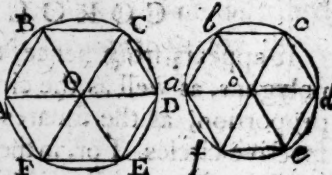
For, by taking away the equal angles OAF, OAB, OBA, OBC, &c. from the equal (right) angles OAP, OAQ, OBQ, OBR, &c. the remainders FAP, BAQ, ABQ, CBR, &c. will also appear to be equal: therefore the triangles FAP, ABQ, BRC, &c. (having also $FA = AB = BC$, &c.) are equal in all respects; and so the angles P, Q, R, &c. as well as the sides PQ, QR, RS, &c. are equal among themselves.

Ax. 4. 1.

PROBLEM XXXI.

Any two circles (ACE, ace) being given, to describe a polygon in, or about the one (ace) that shall be similar to any polygon described in, or about the other (ACE.)

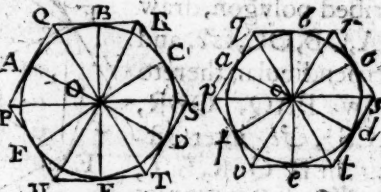
First, having drawn the radii OA, OB, &c. to the angles of the given insc. polyg. ABCDEF, make, at the center o, the angle $ao b = AOB$,



$bo c = BOC$, &c. Then, the chords ab, bc, cd , &c. being drawn, I say, the polygon $abcdef$ will be similar

lar to the given one $ABCDEF$. For, the triangles
^b Cor. 1. AOB, aob ; BOC, boc ; &c. being equiangular^b, the
 to 10. 1. angles ABC, abc must also be equal^c; and^d $AB : ab$
 and 12. 1. $(:: OB : ob) :: BC : bc$. In the same manner, the
^e Ax. 4. 1. other corresponding angles are equal, and the sides
^d 14. 4. containing them proportional: Therefore the two
^e Def. 14. polygons are similar^e.

4. Again, having drawn the radii OA, OB, OC , &c. to the points of contact of the given circumscribing polygon $PQRSTU$;



draw likewise the radii oa, ob, oc , &c. making the angle $aob = AOB$, $boc = BOC$, &c. perpendicular to which^e draw pq, qr, rs , &c. so shall the polygon $pqrstv$ be similar to the given one $PQRSTU$. For, the angles OAQ, OBQ, oaq, obq , being all equal^b, and AOB also $= aob$ ¹; the remaining angles AOB, aob of the two quadrilaterals $AOBQ, aobq$ must be equal^b; as must likewise their halves QQB, oqb (for the right-angled triangles OAQ, OBQ , having $OA = OB$, and OQ common, have also $QQA = QQB$ ¹). In the same manner is $ORB = orb$, &c. whence, the triangles POQ, poq ; QOR, qor , &c. being equiangular, it follows^m that $PQ : pq (:: OQ : oq) :: QR : qr$: And so of the rest. Therefore $PQRSTU$ and $pqrstv$ are similarⁿ.

COROLLARY.

It appears from hence, that the similar inscribed polygons, as well as the circumscribing ones, are in proportion, as the squares of the radii of their respective circles. For, in the former case, $AO^2 : ao^2 ::$

^o $AB^2 : ab^2 ::$ ^r $ABCDEF : abcdef$; and, in the latter, ^o $AO^2 : ao^2 ::$ ^r $PO^2 : po^2 ::$ $PQ^2 : pq^2 ::$ ^r $PQRSTU : pqrstv$.

The End of the FIFTH BOOK.

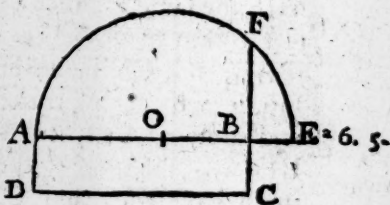
ELEMENTS OF GEOMETRY.

BOOK VI.

PROBLEM I.

To make a square equal to a given rectangle (ABCD).

IN one side AB of the rectangle, produced, take BE = the other side BC; bisect AE in O; and from the center O, at the distance of OA, or OE, let a semi-circle AFE be described; and let CB be produced to meet the circumference thereof in F; then a square described on BF (by 10. 5.) will be equal to the given rectangle ABCD^b.



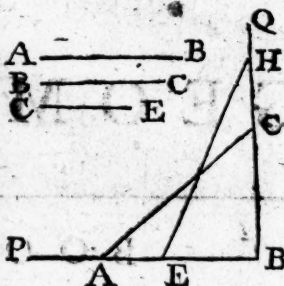
^b Cor. to
19. 4.

PROBLEM II.

To make a square equal to the sum of two given squares.

Let AB and BC be the sides of the two given squares.

Draw two indefinite lines BP, BQ, at right angles to each other^c; in which take BA = BA, BC = BC, and join A, C; ^{c 3. 5.} then a square described on AC (by 10. 5.) will be equal to the sum of the two squares described upon AB and BC^d. ^{d 8. 2.}



SCHOLIUM.

In the same manner a square may be made equal to the sum of three, or more, given squares: for if AB, BC, CE be taken as the sides of the given squares, then, by making BH = AC, BE = CE, and drawing EH, it is evident that a square upon EH will be equal to the sum of the three squares upon AB, BC, and CE; or that, $EH^2 = BH^2 (AC^2) + BE^2 = AB^2 + BC^2 + CE^2$.

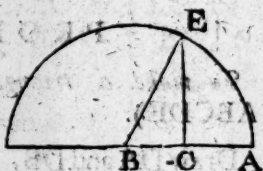
PROBLEM III.

To make a square equal to the difference of two given squares.

Let

Let AB and BC (taken in the same strait line) be equal to the sides of the two given squares.

Upon the center B , with the radius BA , let a circle be described, and make CE perpendicular to BC ; meeting the circumference thereof in E : so shall a square described on CE (by 10. 5.) be equal to BE^2 (BA^2) — BC^2 .

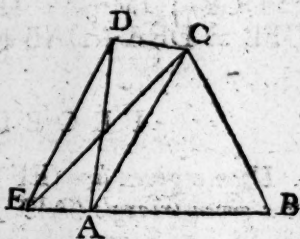


3. 5.
Cor. 1.
8. 2.

PROBLEM IV.

To make a triangle equal to a given quadrilateral (ABCD.)

Draw the diagonal AC , also draw DE parallel to AC , meeting BA produced in E , and join CE ; then will the triangle BCE = the given quadrilateral $ABCD$.



9. 5.

For, the triangles ACE , ACD , being upon the same base AC , and between the same parallels AC and ED , are equal^b; therefore, if ABC be added to each, then also will BCE = $ABCD$ ^c.

Cor. 1.
to 2. 2.
Ax. 4. 1.

PROBLEM II.

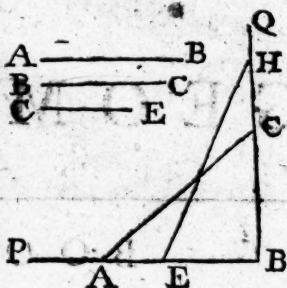
To make a square equal to the sum of two given squares.

Let AB and BC be the sides of the two given squares.

Draw two indefinite lines BP, BQ, at right-angles to each other^c; in

which take BA = BA, BC = BC, and join A, C; then a square described on AC (by 10. 5.) will

be equal to the sum of the two squares described upon AB and BC^d.



SCHOLIUM.

In the same manner a square may be made equal to the sum of three, or more, given squares: for if AB, BC, CE be taken as the sides of the given square, then, by making BH = AC, BE = CE, and drawing EH, it is evident that a square upon EH will be equal to the sum of the three squares upon AB, BC, and CE; or that, $EH^2 = BH^2 (AC^2) + BE^2 = AB^2 + BC^2 + CE^2$.

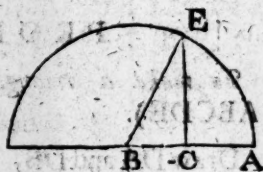
PROBLEM III.

To make a square equal to the difference of two given squares.

Let

Let AB and BC (taken in the same strait line) be equal to the sides of the two given squares.

Upon the center B, with the radius BA, let a circle be described, and make CE perpendicular to BC^s, meeting the circumference thereof in E: so shall a square described on CE (by 10. 5.) be equal to BE² (BA²) - BC².

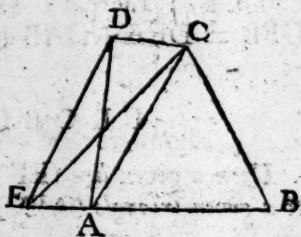


3. 5.
Cor. 1.
8. 2.

PROBLEM IV.

To make a triangle equal to a given quadrilateral (ABCD.)

Draw the diagonal AC, also draw DE parallel to AC^s, meeting BA produced in E, and join CE; then will the triangle BCE = the given quadrilateral ABCD.



9. 5.

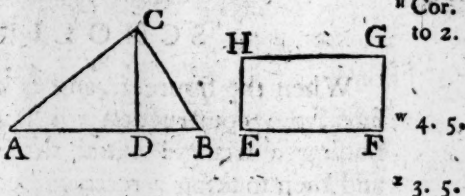
For, the triangles ACE, ACD, being upon the same base AC, and between the same parallels AC and ED, are equal^b; therefore, if ABC be added to each, then also will BCE = ABCD^c.

Cor. 1.
to 2. 2.
Ax. 4^t.

For, it is evident, that LJ , IO and LO are all rectangles¹: therefore $IN = BM^2 = EF^3$, and¹ $IO = LI^4 = ABC^5$.
¹ Cor. to 24. 1.
² 24. 1.
³ Constr.
⁴ 3. 2.
⁵ Cor. 2. to 2. 2.

The same otherwise.

From the vertex C , upon the base AB , let fall the perpendicular CD^6 ; make EH perpendicular to EF^7 , and



equal to a fourth proportional to $2EF$, AB , and CD^8 : then the rectangle EG contained under EF^9 and EH will be equal to the triangle ABC .
⁶ 3. 5.
⁷ 13. 5.
⁸ 10. 4.
⁹ Cor. 2. to 2. 2.

For, since, by construction, $2EF : AB :: CD : EH$, therefore is $2EF \times EH = AB \times CD^2$, and consequently $EF \times EH = \frac{1}{2} AB \times CD = ABC^3$.
² 10. 4.
³ Cor. 2. to 2. 2.

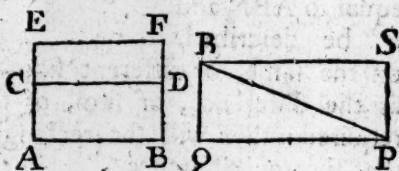
SCHOLIUM.

By either of the two preceding methods, a parallelogram having a given angle may be described upon a given line, equal to a given triangle; if, instead of MBP , MLQ , or BDC , FEH being right-angles, you make them all equal to the angle given: the rest of the construction being the same.

PROBLEM VII.

Upon a given line (AB) to describe a rectangle equal to a given right-lined figure (PQRS).

Let the given figure be divided into triang. PQR , PRS : and upon the given line AB (by the precedent)



let a rectangle $ABDC$, equal to the triangle PQR ,
_{bc}

be described; also upon CD make the rectangle CDFE equal to the triangle PRS: so shall AB EF, which is a rectangle (because both ACE, and BDE are continued right-lines^b) be = PQR + PRS^c = PQRS; which was to be done.

^b 2. 1.

^c Ax. 3.
and 4. 1.

SCHOLIUM.

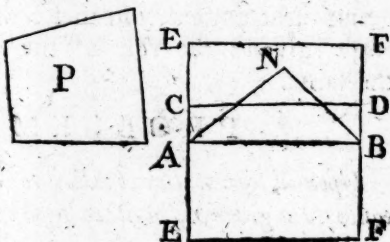
When the figure given has not more than five sides, the construction will be more easy, by first finding a triangle equal to it (by Prob. 4. or 5.) and then making a rectangle equal to that triangle. But if the figure be a rectangle, the easiest way of all, will be to take a fourth-proportional BF to the given line AB and the two sides PQ and PS of the given rectangle (by 13. 5.); which fourth-proportional will be the altitude of the rectangle required. For, since $AB : PQ :: PS : BF$ (by Constr.) therefore (by 10. 4.) $AB \times BF = PQ \times PS$.

PROBLEM VIII.

To describe a rectangle equal to the sum, or difference of two given right-lined figures.

Let the two given figures be ABN and P.

By the precedent, let two rectangles AD and AF, respectively equal to ABN and P, be described,



on the same, or different sides of AB, according as the difference, or sum, of the two figures is required: then will the rectangle CF be equal to that sum or difference^c.

^c Ax. 4.
or 5. 1.

COROL.

COROLLARY.

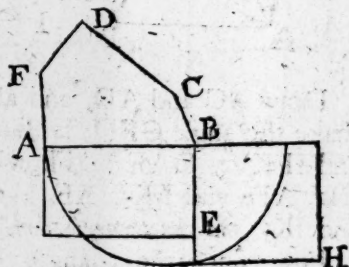
Hence two lines having the same ratio with two given right-lined figures, are determined: for $AC : AE :: AD (ABN) : AF^f (P)$.

^f 7. 4.

PROBLEM IX.

To make a square equal to any right-lined figure given (ABCDF).

Upon AB describe a rectangle AE equal to ABCD^a; then make a square BH equal to that rectangle^b, and the thing is done.



^a 7. 6.

^b 1. 6.

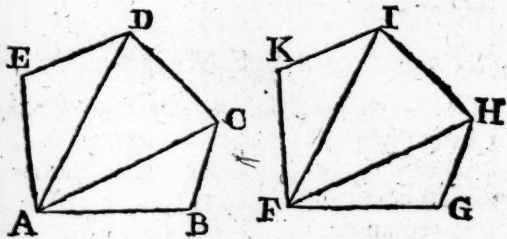
SCHOLIUM.

After the same manner (from prob. 8.) a square may be described equal to the sum, or difference, of any two given right-lined figures.

PRO-

PROBLEM X.

To describe a figure (FGHIK) equal, and similar to a given right-lined figure (ABCDE).



¹ 2. 5. Draw AC and AD, and also FG equal to AB¹;
² 7. 5. make the angle GFH = BAC², HFI = CAD,
 and IFK = DAE²; likewise make FH = AC,
 FI = AD, and FK = AE¹; then draw GH, HI,
 and IK, and the thing is done.

¹ Ax. 10. For, since the triangle FGH = ABC¹ and
¹ FHI = ACD¹, &c.; therefore is the whole po-
 lygon FGHIK, also, equal to the whole polygon

² Ax. 4. 1. ABCDE².

Moreover, these equal triangles being also equi-
 angular¹, it is manifest, that G = B, GHI = BCD²,
 HIK = CDE, and so on; therefore, FG being also
 = AB, GH = BC, HI = CD¹, &c. the two
 polygons ABCDE, FGHIK are similar to each

² Def. 14. other².

4.

SCHOLIUM.

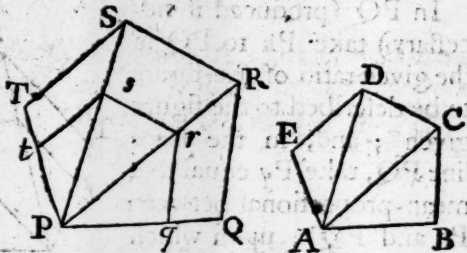
The figure FGHIK may be otherwise con-
 structed, by making the triangles FGH, FHI, &c.,
 respectively equilateral to ABC, ACD, &c. as is
 evident from 14. 1. and Ax. 4.

P R O.

PROBLEM XI.

Upon a given line (AB) to describe a figure (ABCDE) similar to a given right-lined figure (PQRST).

Draw PR and PS, and in PQ (produced if needful) take $Pq = AB$; draw qr parallel to QR , meeting PR in r ;



also draw rs and st , parallel to RS and ST , intersecting PS and PT in s and t ; then upon AB, by the precedent, describe a polygon equal and similar to $Pqrst$; and the thing is done.

For, since any angle BCD (qrs) of the polygon ABCDE, is equal to its correspondent QRS ; and also CB (qr) : CD (rs) :: RQ : RS , therefore the two polygons ABCDE, PQRST are like to each other.
P Cor. 1. to 7. 1. 21. 4. Def. 14. of 4.

SCHOLIUM.

This last problem may be otherwise constructed by making the triangles ABC, ACD, ADE equiangular to the triangles PQR, PRS, PST, respectively.

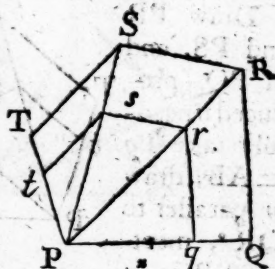
For, then the angle BCD being = QRS , CDE = RST , &c.; and also BC : QR (:: AC : PR) :: CD : RS , &c. the two polygons must therefore be similar to each other.
Ax. 4. 1. 14. 4. Def. 14. of 4.

PRO-

PROBLEM XII.

To describe a figure, similar to a right-lined figure given (PQRST), which shall be to it in a given ratio of one right-line to another.

- In PQ (produced if necessary) take Pn to PQ in the given ratio of the figure to be described to the figure given^{12.5.}; and, in the same line PQ, take Pq equal to a mean-proportional between Pn and PQ^{14.5.}; upon which (by the precedent) let Pqrst, similar to PQRST, be described, and the thing is done.

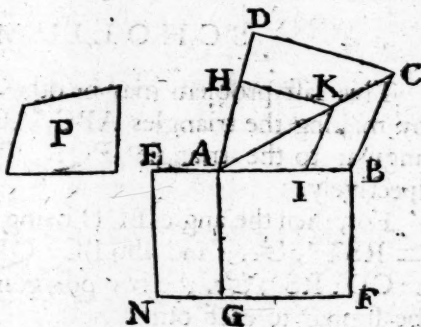


- For, since $Pn : Pq :: Pq : PQ$ (by Constr.); therefore $Pn : PQ :: Pqrst : PQRST$ ^{27.4.}.

PROBLEM XIII.

To describe a figure that shall be equal to one right-lined figure given (P), and similar to another (ABCD).

- Upon AB make the rectangle ABFG^{7.6.} $\equiv ABCD$ ^{2.}, and upon AG make the rectangle AGNE $\equiv P$ ^{2.}; in AB take AI equal to a mean proportional between AB and



- AE^{14.5.}; and upon AI let AIKH be described similar,

ELEMENTS

OF

GEOMETRY.

BOOK VII.

DEFINITIONS.

1. **A** Right-line is said to be a perpendicular to a plane, when it is perpendicular to all right-lines, that can be drawn in that plane, from the point on which it insits.

2. One plane is said to be perpendicular to another, when all right-lines drawn in the one, perpendicular to the common section, are perpendicular to the other.

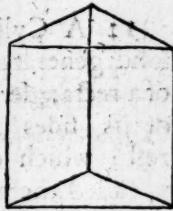
3. Parallel planes are those, which are every where equally distant, the one from the other.

4. A Solid is that, which has length, breadth, and thickness.

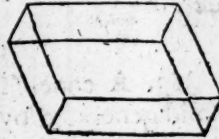
5. Similar solids are such, as are bounded by an equal number of similar planes.

6. A Prism

6. A Prism is a solid, whereof the planes of the sides are parallelograms, and whereof the two ends, or opposite bases, are plane, rectilinear figures, parallel to each other.

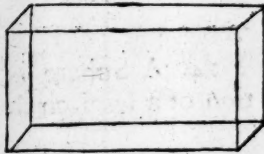


7. A parallelepipedon is a solid bounded by six parallelograms, whereof the opposite ones are parallel, equal, and like to each other.

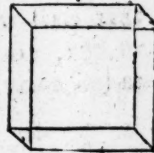


8. An upright prism, or parallelepipedon, is that, whereof the planes of the sides are perpendicular to the plane of the base.

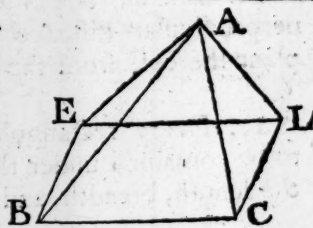
9. A rectangular parallelepipedon is that, whose bounding planes are all right-angles, and which stand at right-angles one to another.



10. When all the bounding planes are squares, the parallelepipedon is called a cube.



11. A Pyramid is a solid, whose base is any right lined plane figure, and whose sides are triangles, having all their vertices united in a point, above the base, called the vertex of the pyramid. Thus

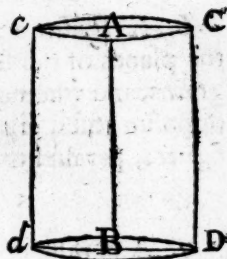


ABCLE represents a pyramid, whose vertex is A, and base BCLE.

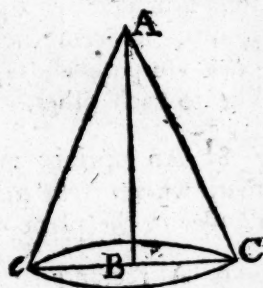
K

12. A

12. A Cylinder ($DCcd$) is a solid generated by the rotation of a rectangle $ACDB$ about one of its sides AB , supposed at rest; which quiescent side AB is called the axis of the cylinder.



13. A cone (ACc) is a solid generated by the rotation of a right-angled triangle ABC about its perpendicular AB , called the axis of the cone.



14. A Sphere is a solid generated by the rotation of a semi-circle about its diameter.

15. The Frustum of a pyramid, or cone, is that part which remains, when any part next the vertex, cut off by a plane parallel to the base, is taken away.

16. The altitude of a pyramid, or prism, is the perpendicular distance of the vertex, or upper plane thereof, from the plane of the base.

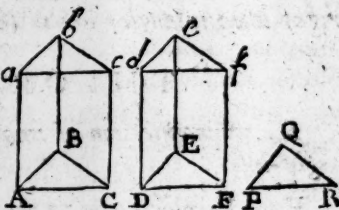
17. Every rectangular parallelepipedon is said to be contained under the three right-lines that are the length, breadth, and altitude thereof.

18. A Plane is said to be extended (or to pass) by a right-line, when every part of the latter is placed in, or touched by, the former.

An AXIOM.

Upright prisms ($AabcCBA$, $DdefFED$) of the same altitude, standing upon bases (ABC , DEF) equal and like to each other, are themselves equal.

To see the evidence of this AXIOM in the strongest light, conceive a right-lined plane figure PQR to be formed, equal and like in all respects to the bases ABC , DEF of the two prisms;

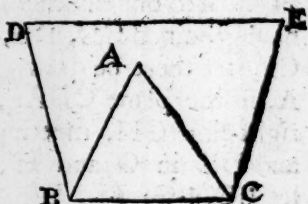


upon which, conceive the prisms to be placed, one after another, so that their bases may coincide therewith. Then, because the planes of the sides stand, in both cases, perpendicular to the plane of the base, upon the same lines PQ , QR , PR , and are carried up to the same height, it is manifest, that the bounds of the two solids, when thus placed, have the very same position; and, consequently, that the solids themselves, occupying (successively) the same identical space, are equal the one to the other.

A POSTULATE.

That by any two right-lines (AB , AC) meeting in a point, a plane may be extended.

In order the better to comprehend the sense and design of this Postulate, let a plane $BDEC$, extended by the right-line joining the points B and C , be conceived to be revolved



about upon that line, till it meets with, or takes in, the point A ; then the plane including, in that position, all

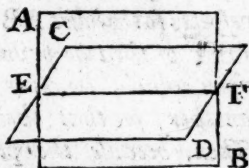
the three points B, C, and A, it also includes, or is extended by, the right-lines AB, AC, BC, joining those points; which are in the same plane with their extremes (by def. 6. 1.)

Hence it appears, that, by any three points, a plane may be extended; and that all the three sides of any right-lined triangle, are in the same plane.

THEOREM I.

The common section of two planes (AB, CD) is a right-line.

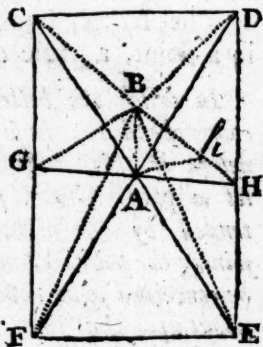
For, between the two extreme points E, F of the common section, let a right-line EF be drawn^a; then, that line being in the plane AB^b, and also in the plane CD^b, it must, of consequence, be the common section of them both.



THEOREM II.

If a right-line (AB) be perpendicular to two other right-lines (CE, DF) cutting each other, at the common section (A), it will be perpendicular to the plane (CDEF) passing by those two lines.

Take AC, AD, AE, AF all equal to one another; and, having joined CD, DE, EF, CF, let there be drawn thro' A, in the plane CDEF, any right-line GH, meeting CF and DE in G and H; and let BC, BG, BF, BD, BH, and BE be also joined,



Because

Because $AC^b = AE = AD = AF$, and $CAF =$ ^b Confr. DAE^c , therefore is $CF = DE^d$, and the angle FCA^e ^{3. 1.} (or GCA) $= DEA$ or (HEA) ; and so, GAC be- ^d Ax. 10. 1. ing likewise $= HAE^e$ and $AC = AE^b$, thence will $AG = AH^e$ and $GC = HE$. ^e 15. 1.

Again, since the right-angled triangles CAB , DAB , EAB , FAB have their bases all equal ^f, and ^f Hyp. the perpendicular AB common, their hypotenuses BC^d , BD , BE , BF will be equal too; and therefore, the triangles CBF , DBE being mutually equilateral, the angle FCB (or GCB) must be $= DEB$ (or HEB^g); whence, GC being also $= HE$, and $BC =$ ^g 14. 1. BE , thence is $BG = BH^d$: Therefore, AG being likewise (as is proved above) $= AH$, and AB common, the angles GAB , HAB are equal, and consequently right-angles ^h. In the same manner, AB ^h Def 8. of 1. is perpendicular to every other right-line drawn thro' A in the plane $CDEF$; which was to be demonstratedⁱ. ⁱ Def. 1. of 7.

COROLLARY.

Hence it will appear, that, if one right-line (AB) , meeting several others $(AF, AE, \&c.)$ in the same point (A) , is perpendicular to them all; these last will be all in the same plane. Because it is impossible for a right-line (Ab) drawn from A , out of the plane $(FEDC)$ of the two former of these, to be perpendicular to AB ; seeing the angle BAb is less, or greater, than a right-angle (or BAH^k), according as Ab is posited above, or be- ^k 2. 7. low the said plane $FEDC^l$. ^l Ax. 2. 1.

THEOREM III.

If thro' any given point (A) in a given plane (BCD) , a line (CD) be drawn, and perpendicular to that line, at the same point (A) , two other lines (AB, AE) be also drawn, the one (AB) in the plane given (BCD) , and the other in any other plane (CDE)

K 3

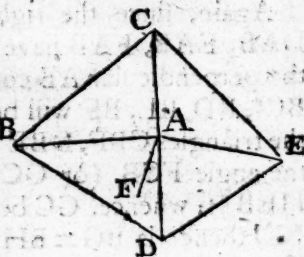
passing

passing by the first line (CD); then, I say, that a right-line (AF) drawn from the given point (A), at right-angles to the first perpendicular (AB) in the plane (BAE) of the two, will be perpendicular to the given plane (BCD) at the given point (A).

ⁿ Hyp.

^s 2. 7.

For CA being perpendicular both to AB and AEⁿ, it will likewise be perpendicular to AFⁿ; B and so FA, being perpendicular to AB^m (as well as to CA) is also perpendicular to the plane BCD, in which AB and CA are drawnⁿ.



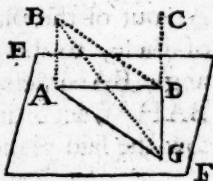
SCHOLIUM.

In this last Theorem, the manner of erecting a perpendicular to a plane, at a point given, is indicated, and the consistence of the first definition, of this book, evinced.

THEOREM IV.

Two right-lines (AB, CD), perpendicular to the same plane (EF), are parallel to each other.

Draw in the plane EF, the right-line AD, and also DG perpendicular to AD; make DG = AB, and let AG, BG, and BD be drawn.



The triangles BAD, ADG, having AB = DG^o, AD com-
^o Confir. mon, and the angle BAD (= right-angle^p) =
^p Def. 1. 7. ADG^o, will also have BD = AG^o; And so, the
^o Ax. 10. 1. triangles BDG, BAG being mutually equilateral,
^o 14. 1. the angle^r BDG must be = BAG = ^p right-angle;
 But the line CD (as well as BD and AD) being per-
 pendicular

In the plane ED, from any point K, draw KI perpendicular to the common section CD. Then, the angle DIK being a right-angle = DQP^a, IK will be parallel to PQ^a, and therefore perpendicular to the plane AB^a. By the same inference, all other right-lines drawn in the plane ED, perpendicular to the common section CD, are also perpendicular to the plane AB. Therefore the plane ED itself is perpendicular to the plane AB^b.



^a Def. 1. 7.

^a 4. 1.

^a 5. 7.

^b Def. 2. 7.

COROLLARY I.

Hence it will appear, that the plane AB (according to the sense of the definition) is perpendicular to the plane ED: For a right-line QR drawn in the former, perpendicular to the common section CD, being also^c perpendicular to PQ, it^d (and consequently the plane AB in which it is) will be perpendicular to the plane ED^e.

^c Hyp. and

Def. 1. 7.

^d 2. 7.

^e 6. 7.

COROLLARY II.

Hence it also appears, that a line standing at right-angles to one of two perpendicular planes at any point (I) in the common section (CD) must be in the other plane: For the line IK, in the plane ED, is perpendicular to the plane AB; besides which line, another perpendicular to AB, from the same point I, cannot be drawn^f.

^f Cor. to

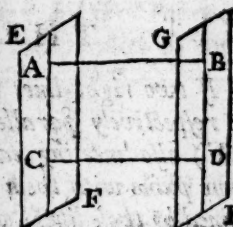
4. 7.

THEOREM VII.

Planes (EF, GH) to which one and the same right-line (AB) is perpendicular, are parallel to each other.

From

From any point C, in the plane EF, let CD be drawn parallel to AB; which (as well as AB) will be perpendicular to both the planes^a; and so the angles A, B, C, D (when AC and BD are joined) being all right-ones^b, the figure ABDC (whereof the sides AC, BD are in the same plane with the parallels AB, CD^c) will therefore be a rectangular parallelogram^k; and consequently $CD = AB$ ^l. By the very same argument, all other perpendiculars, terminated by the two planes, are equal among themselves; *which was to be demonstrated*^m.



^a 5. 7.

^b Def. 1.
7.

^c Def. 6. 1.
^k 4. 1. and
Def. 24.
24. 1.

^m Def. 3.

COROLLARY.

Hence, all right-lines perpendicular to one of two parallel planes, are also perpendicular to the other.

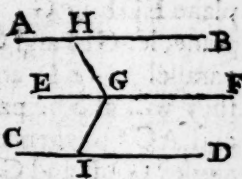
SCHOLIUM.

From the two last Theorems, the sense, and propriety of the two definitions of perpendicular, and parallel planes, appear manifest.

THEOREM VIII.

Right-lines (AB, CD) parallel to one and the same right-line (EF), tho' not in the same plane with it, are also parallel to each other.

Let GH and GI be drawn perpendicular to EF, in the planes AF and ED of the proposed parallels. Thenⁿ shall GF be perpendicular to the plane passing by HGI; and HB, ID will also be perpendicular to the same plane^o, and therefore parallel to each other^p.



ⁿ 2. 7.

^o 5. 7.

THEO-^p 4. 7.

THEOREM IX.

If two right-lines (AB, AC) meeting each other, be respectively parallel to two other right-lines (DE, DF) also meeting each other, and not being in the same plane with them; the angles (BAC, EDF) contained by those lines, will be equal.

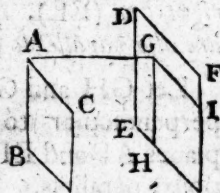
Take AB, AC, DE, DF all equal to each other, and let BE, AD, CF, BC, EF be drawn. Then AB and ED, as well as AC and DF, being equal, and parallel¹, BE and CF will be both equal, and parallel, to AD², and therefore equal, and parallel, to each other³; whence BC is also equal to EF⁴; and so, the triangles ABC, DEF being mutually equilateral, the angles BAC, EDF are likewise equal⁵.



THEOREM X.

If two right-lines (AB, AC) meeting each other, be respectively parallel to two other right-lines (DE, DF) also meeting each other, and not being in the same plane with them, the planes (BAC, EDF) extended by those lines, will be parallels.

Let AG be perpendicular to the plane BAC, meeting the plane EDF in G; in which last plane, let GH and GI be drawn parallel to ED and DF; and they will also be parallel to AB and AC¹; whence, seeing the



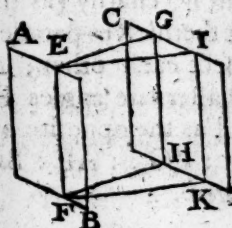
angles GAB and GAC are both right², AGH and AGI must likewise be right-angles³; and so AG being

being perpendicular to the plane EDF^a (as well^a 2.7. as to BAC^b), the two planes are parallel to each^b Constr. other^c: 7.7.

THEOREM XI.

The sections (EF, GH) made by a plane (EFHG) cutting two parallel planes (AB, CD), are also parallel, the one to the other.

Let EG and FH be drawn parallel to each other, in the plane EFHG; also let EI, FK be perpendicular to the plane CD, and let IG, KH be joined: Then, EG being parallel to FH^d, and EI to FK^e, the angle GEI is = HFK^f; but the angle EIG is also = FKH, being both right-angles^g; and EI is = FK^h: Therefore EG will be equalⁱ (as well as parallel) to FH; and consequently EF likewise parallel to GH^k.



^d Constr.

^e 4.7.

^f 9.7.

^g Def. 1.7.

^h Def. 3.7.

ⁱ 15. 1.

^k 26. 1.

COROLLARY.

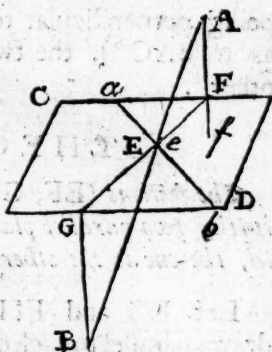
It appears from hence, that parallel lines, terminated by the same parallel planes, are equal to each other.

THEOREM XII.

If, from the two extremes of a right-line (AB) cutting a plane (CD), two perpendiculars (AF, BG) be drawn to the plane; the right-line (FG) joining the points where they meet the plane, will pass thro' the point (E) in which the proposed line (AB) cuts the plane, so as to be divided by it into two parts (FE, EG), having the same ratio to each other as those two perpendiculars (AF, BG).

For,

- For, if AF be produced to *f*, the lines Af and BG (which are both perpendicular to the plane CD¹) will be parallel to each other^m; therefore AB and FG being both in the same plane with these parallels, in which their extremes are positedⁿ, they must necessarily (as they are not themselves parallels) intersect each other; And so the alternate angles FAE, GBE being equal^o, as well as the opposite ones FEA, GEB^p, thence will $FE : EG :: AF : BG$ ^q; which was to be demonstrated.



COROLLARY.

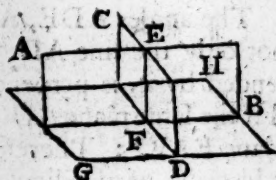
- Hence, if in the plane CD, the lines FC, GD be made parallel, the one to the other, and in them be taken $Fa = FA$, and $Gb = GB$; then will the line (*ab*) joining the points *a* and *b*, cut FG in the very same point in which it is cut by AB. For, if *e* be taken as the intersection of *ab* and FG, the triangles *aFe*, *Gbe* will be equiangular^r; whence $Fe : eG :: Fa (FA) : Gb (GB) :: FE : EG$ ^s. Therefore, seeing FG is divided in one and the same ratio, both by *e* and E, these points must necessarily coincide^t.

THEOREM XIII.

If two planes (AB, CD) cutting each other, be both perpendicular to a third plane (GH), their common section will also be perpendicular to the same plane (GH).

For,

For, from the extreme point F of the common section, let the right-line FE be erected perpendicular to the plane GH: which line being in both the planes AB, CD¹, it must necessarily be their common section. Therefore the common section is perpendicular to the plane GH^m.

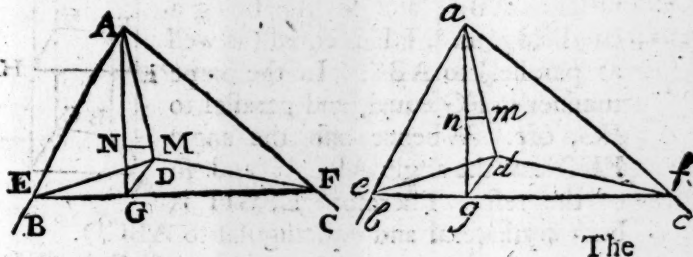


¹ Cor. 2.
to 6. 7.
^m Constr.

THEOREM XIV.

If, from the angular points (A, a) of two equal angles (BAC, bac) two right-lines (AD, ad) be drawn, or elevated on high, above the planes of the said angles, so as to form equal angles with the lines first given, each to its correspondent ($\angle DAB = \angle dab$, $\angle DAC = \angle dac$), and if, from any points (M, m) in those elevated lines, perpendiculars (MN, mn) be let fall upon the planes (BAC, bac) of the first mentioned angles; these perpendiculars will be, in proportion, as the parts (AM, am) of the elevated lines included between them and the angular points (A, a) first named.

Make AD and ad equal to each other; and in the planes ADB, ADC, adb, adc, draw DE, DF, de, df perpendicular to AD and ad; and from their interfections with AB, AC, ab, ac, draw EF and ef, meeting AN and an (produced) in G and g, and let D, G, and d, g be joined.



The

- ^a Constr. The angles ADE, ADF being both right-ones^a,
^o 2. 7. not only the line AD^o, but the plane ADG ex-
^p 6. 7. tended by it, is perpendicular to the plane EDF^p.
 But the same plane ADG is also perpendicular to the
 plane EAF^p: Therefore the common section EF
^q 13. 7. is likewise perpendicular to the plane ADG^q; and
^r Def. 1. 7. consequently the angle EGA a right-one^r. By the
 very same argument, *ega* is a right-angle. Now the
 triangles ADE, *ade*; ADF, *adf* being equal in all
^s Hyp. and respects^s, and the angle EAF = *eaf*^s, the triangles
^{15. 1.} AEF, *aef* are also equal and alike^u; and so, the
^t Hyp. angle AEG being = *aeg*, EGA = *ega*, and AE =
^v Ax. 10. 1. *ae*, thence is AG = *ag*^w, and the angle DAG
^x 15. 1. (MAN) = *dag* (*man*), because ADG, *adg* are both
^y 16. 1. right-angles^o. Therefore MN : *mn* :: AM : *am*^y.
^z 14. 4.

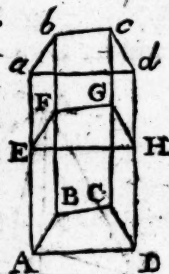
COROLLARY.

Hence the two perpendiculars MN, *mn* subtend equal angles at the points (A, *a*) from whence the two elevated (or inclining) lines are drawn.

THEOREM XV.

If any solid (Ac) having a rectilinear base (ABCD), whereof the planes (Ab, Bc, Cd, AD) of the sides are parallelograms, be cut by a plane parallel to the base, the section (EFGH) will be equal, and similar to the base.

- For, the plane EFGH being paral-
^a Hyp. lel to ABCD^z, EF is therefore pa-
^b 11. 7. rallel to AB^a; and so, AF being a
^c Def. 24. parallelogram^b, EF is equal (as well
 of 1. as parallel) to AB^c. In the same
^d 24. 1. manner is FG equal, and parallel to
^e 9. 7. FG, &c. Whence also the angle
 EFG is = the angle ABC^d; and so
 of the rest. Therefore EFGH is
 both equilateral and equiangular to ABCD.



COROL

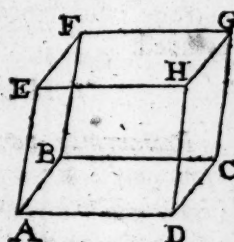
COROLLARY.

Hence the opposite bases of a prism are equal and similar (as well as parallel) to each other.

THEOREM XVI.

If, from one of the angular points (A) of any parallelogram (AC) a right-line (AE) be elevated above the plane of the parallelogram, so as to make any angles (EAB, EAD) with the two contiguous sides thereof, and there be also drawn, from the three remaining angular points, three other right-lines (BF, CG, DH) parallel, and equal to the former (AE); then, the extremes of those lines being joined, I say, the figure (AG) thus described, will be a parallelepipedon.

For AE, BF, CG, DH being all parallel to each other^e, AE and BF are in the same plane^f, as are also AE and DH, &c. Therefore, all these lines being^g equal among themselves, AF, AH, DG, and BG are paral-



^eHyp. and
8. 7.
^fDef. 13.
1.
^gHyp.

lelograms^h; and so, EF being parallel to AB, parallel to DC, parallel to HGⁱ, EF and HG are in the same plane^j; and EG is also a parallelogram^k, equilateral to its opposite AC: but EG is equiangular, and parallel (as well as equilateral) to its opposite AC; because, EF being parallel to AB, and EH to AD, the angle FEH is therefore = BAD^l, and the plane EG parallel to the plane AC^m. And in the same manner, the other opposite parallelograms appear to be equiangular and parallel (as well as equilateral). Therefore the solid AG, bounded by them, is a parallelepipedonⁿ.

COROL.

ⁿDef. 7.
of 7.

COROLLARY.

If the angle A of the parallelogram AC be a right-one, and AE be erected perpendicular to the plane AC; then will the parallelepipedon be a rectangular one: for, all the three contiguous planes AC, AF, AH being rectangular^{24. 1.}; their opposites will be rectangular likewise^{16. 7.}; and so, the angles HGF, HGC being right-ones, HG will be perpendicular to the plane GB^{2. 7.}; and consequently both the planes EG and DG likewise perpendicular to the plane BG^{6. 7.}. And so of the rest.

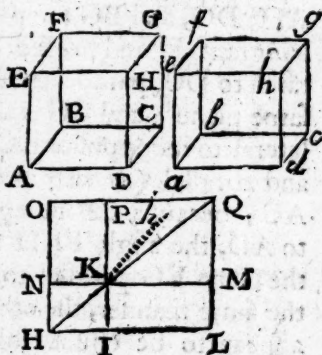
SCHOLIUM.

In this Theorem, a way to describe a parallelepipedon of any given dimensions, is indicated; and the consistence of the 7th and 9th definitions evinced.

THEOREM XVII.

Rectangular parallelepipedons (AG, ag) standing upon equal bases (AC, ac), and having equal altitudes (AE, ae), are equal.

Let the rectangles OK and KL, equal and like to the bases AC and ac of the two solids, be so formed, that NK may be in the same strait line with KM; then shall PK be also in the same strait line with KI^{2. 1.}; and the figures NI, PM, OL, formed by producing the sides of the two rectangles, will likewise be rectangles^{Cor. to 24. 1.}.



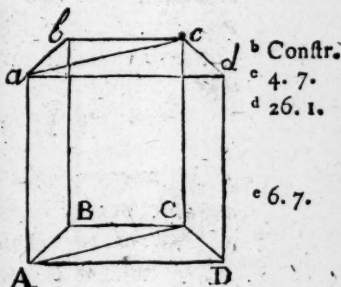
Now

Now, HK and QK being drawn, the triangle NHK = IHK^u, PKQ = MKQ^u, and the rectangle^a 24. 1. OK = LK^u; and consequently (by the addition of^w Hyp. equals) OQKH = LQKH. Therefore, HKQ being a diagonal to the rectangle OL^x, dividing it^x 24. 1. into two equal, and like triangles OQH, LQH^u, and Ax. 2. if upon these, as bases, two upright prisms be conceived to be erected, of the same common altitude (Kb) with the proposed solids, these prisms will also be equal^v. But the former of these is com-^y Ax. 1. 7. posed^z of three prisms, on the bases OPKN, NHK, ^z Cor. to 18. 7. KPQ; and the latter of three others, on KMLI, HIK, KMQ; whereof the second and third, in both ranks, are respectively equal^v. Therefore the remaining two, on the bases OPKN, KMLI must also be^t equal. But the former of these^t Ax. 5. is = AG^v, and the latter = ag: Therefore, also, ^a Ax. 1. is AG = ag².

THEOREM XVIII.

If, at the angular points of any given right-lined figure (ABCD), equal perpendiculars (Aa, Bb, Cc, Dd) be erected to the plane thereof, and the extremes of these (a, b; b, c, &c.) be joined; an upright prism (AabcdDCBA) on the given base (ABCD) will thereby be formed.

For, Aa, Bb, Cc, Dd being all equal^b, and parallel^c, it is evident that AabB, BbcC, &c. are parallelograms^d; and that the planes of these are all perpendicular to that of the base ABCD^e (since Aa, Bb, Cc are so, by construction). Moreover it will appear, that abcd



L

ABCD;

- ABCD; for, the lines ac , AC (when a , c and A , C are joined) being parallels^f (as well as ab , AB ; ad , AD), the plane abc is, therefore, parallel to ABC ^g (or $ABCD$); and acd is likewise parallel to $ABCD$. But abc and acd are in one plane; because Aa being perpendicular to both of them^h, andⁱ consequently to all the lines ab , ac , ad ; these must necessarily be all in one^k plane, parallel to $ABCD$; *which was to be demonstrated*^l.
- ^f 26. 1.
^g 10. 7.
^h Cor. to 7. 7.
ⁱ Def. 1. 7.
^k Cor. to 2. 7.
^l Def. 6.

COROLLARY.

- It appears from hence, that, if upon all the parts ABC , ACD , into which any rectilinear figure $ABCD$ is divided, upright prisms ($AabcCB$, $AacdDC$) of the same altitude be constituted; these prisms will form one prism, on the (whole) given base $ABCD$; seeing that abc and acd form one continued plane superficies $abcd$ ^m, parallel to $ABCD$.
- ^m 18. 7.

SCHOLIUM.

- After the same way, a prism, any how inclining on the given base $ABCD$, may be constructed; by giving to Aa the proposed inclination, and then drawing Bb , Cc , Dd parallel, and equal thereto. For $AabB$, $BbcC$, &c. will (still) be parallelograms^f: And, that $abcd$ is one plane, parallel to $ABCD$, will also appear (*in the same manner*); if a perpendicular from a to the plane $ABCD$, be conceived to be drawn.
26. 1.

THEOREM XIX.

If, on equal bases (ABC , $PQRS$), an upright triangular prism, and a rectangular parallelepipedon be erected, of the same altitude; the two solids, themselves, will be equal.

Let

Let CD be perpendicular to AB, and let the rectangles ADCE, BDCF be completed; also let GH be drawn



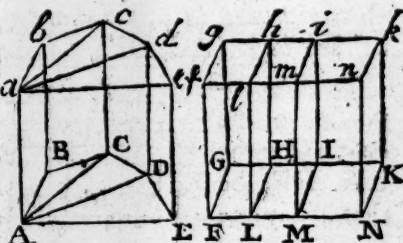
parallel to AE, bisecting AB in G; so shall $AGHE = \frac{1}{2}ABFE^b = \frac{1}{2}ACB^c = PQRS^d$. Now, the two prisms on ADC^c and BDC, into which, *that* on ABC may be divided^e, will be respectively equal to two others, on the equal and similar^f bases AEC and BFC^g; and consequently the prism on ACB = half the prism on AEFB^h = half the two prisms on AGHE and BGHFⁱ = the prism on AGHE = the prism on^k PQRS^{*}.

^b 1. 2.
^c Cor. to 2. 2.
^d Hyp.
^e Cor. to 18. 7.
^f Ax. 24. 1.
^g Ax. 1. 7.
^h Ax. 4. 1.
ⁱ Ax. 3. 1.
^k 17. 7.

THEOREM XX.

Every upright prism (AaceA) is equal to a rectangular parallelepipedon (Fk) of equal base, and altitude.

Let AC, ac, AD, ad be drawn; and in the base FK of the parallelepipedon, let HL, IM, be drawn parallel to FG, in such fort that the rectangles FH, HM, IN may be respectively equal to the triangles ABC, ACD, ADE^k: Then also shall the prism (AabcCBA)^{*} 6. 6.



* In this Theorem, the representations of the prisms are not described; because a great multiplicity of lines, tends to produce confusion in the mind of a learner; especially where solids are represented. The schemes, however, may be formed, at large, by those who think proper to do it: But every tittle of the demonstration will remain the same.

- ¹ 19. 7. on ABC, be equal to the parallelepipedon (Fb) on FH¹; and the prism (AacdDCA) on ACD, equal to the parallelepipedon (Li) on LI¹; and consequently the whole prism (AaceA) on ABCDE, equal to all the parallelepipedons on FK, which form one parallelepipedon (Fk); because Li, Mm are in the plane Fn^m, and Hb, li in the plane Gk^m.

^m 18. 7.
and Cor.
2. to 6. 7.

COROLLARY.

Hence all upright prisms, having equal bases, and altitudes, are equal among themselves.

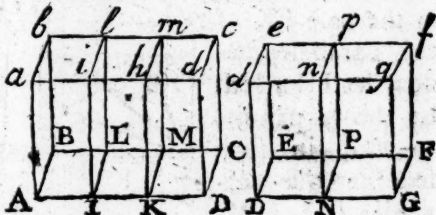
SCHOLIUM.

In the very same manner, the aggregate of any number of prisms, of one common altitude, will appear to be equal to one single prism, or parallelepipedon, of the same altitude, whose base is equal to the sum of all theirs.

THEOREM XXI.

Rectangular parallelepipedons (Ac, Df) having equal altitudes (Aa, Dd) are in the same proportion as their bases (AC, DF).

Let the proportion of the base AC to the base DF, be that of any one number m (3) to any other number n (2).



Let AC be divided into m (3) equal parts (or rectangles) AL, IM, KC (by dividing AD into that number of equal parts^m, and drawing IL, KM parallel to AB)ⁿ: And let DF be divided, in like manner, into n (2) equal parts, or rectangles, DP, NF: Which parts, taken singly, will be equal, in magnitude, to those of the former division^o; and so the

^m 11. 5.

ⁿ 9. 5. and

¹ 2.

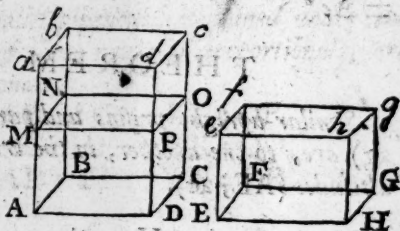
^o Hyp. and
Ax. 8. 4.

the parallelepipedons upon them (Al, Im, Kc, Dp, Nf) will likewise be all equal: Therefore the solid Ac is in proportion to the solid Df , as the number of parts in Ac to the number of equal parts in Df , or as the number of parts in AC to the number of equal parts in DF , that is, as AC to DF .—If the bases are supposed to be *incommensurable*, the solids will still be in the same ratio with them, as appears from the reasoning laid down in the *Scholium* to *Theor. VII. Book IV.*; which is equally applicable in this case.

THEOREM XXII.

Rectangular parallelepipedons (Ac, Eg) standing upon equal bases (AC, EG) have the same ratio as their altitudes (Aa, Ee).

Let AO be a parallelepipedon on the base AC , whereof the altitude AM is equal to that (Ee) of the parallelepipedon Eg : So shall



the solid $AO =$ the solid Eg . (But if Ab and AN be considered as bases) it will be $Ac : AO$ (or Eg) :: $Ab : AN$:: $Aa : AM$ (or Ee): which was to be proved.

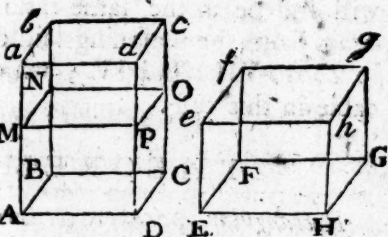
COROLLARY.

Hence, and from the preceding Theorem, it follows, that all upright prisms are, also, as the bases, when the altitudes are equal; and as the altitudes, when the bases are equal; all such solids being (by *Theor. XX.*) equal to rectangular parallelepipedons of equal base and altitude.

THEOREM XXIII.

Upright prisms and parallelepipeds (Ac, Eg) which have their bases and altitudes reciprocally proportional (AC : EG :: Ec : Aa), are equal to each other.

Let AO be a prism on the base AC, whereof the altitude AM is equal to that (Ee) of the prism Eg.

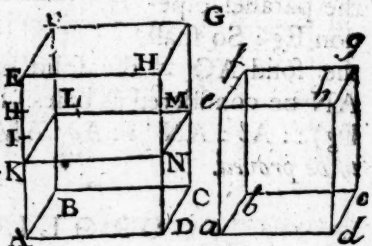


- Then $AO : Eg :: A$
^w 21. 7. $AC : EG :: Ee$
^x Hyp. $(AM) : Aa :: AO : Ac$; and consequently Eg
^y 22. 7. $= Ac^2$
^z Ax. 4. of 4.

THEOREM XXIV.

Similar upright prisms and parallelepipeds (AG, ag) are, to one another, in the triplicate ratio of their altitudes (AE, ae).

Having made $AH = ae$, take AE, AH, AI, AK in continued proportion^a; and let AM be a prism on the base AC , whereof the altitude is AK . Then (because of the similar planes) it will



- be $AC : ac :: AB^2 : ab^2 :: AE^2 : ae^2 (AH^2)$
^b 26. 4. $:: AH^2 :: AI^2 :: AH (ae) : AK^2$; and so, the
^c Cor. to $bases and altitudes of the solids AM, ag being re-$
^{11. 4.} $ciprocally proportional, the solids themselves are$
^{27. 4.} equal;

equal^f; and therefore, $AG : ag :: AG : AM^e ::$ ^{23. 7.}
 $AE : AK^b$: But the ratio of AE to AK is tri- ^{5 Ax. 1.}
 plicate to that of AE to ¹AH (or *ae*). There- ^{of 4.}
 fore, &c. ^{22. 7.}
^{1 Def. 7.}
^{of 4.}

COROLLARY I.

Hence, cubes are in the triplicate ratio of their sides, or altitudes.

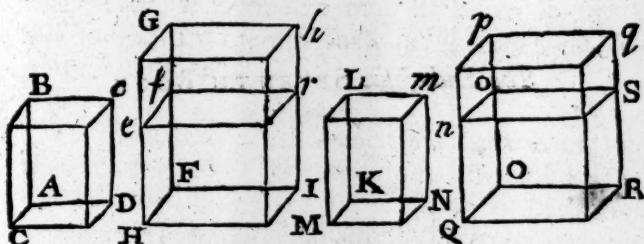
COROLLARY II.

Hence, also, all similar upright prisms, are to one another, as the cubes of their altitudes; since both prisms and cubes, are in the *same* triplicate ratio of the altitudes.

THEOREM XXV.

Rectangular parallelepipeds, contained under the corresponding lines of three ranks of proportionals, are themselves proportionals.

I say, if $\begin{cases} AB : FG :: KL : OP, \\ AC : HF :: MK : QO, \\ AD : FI :: KN : OR, \end{cases}$



then shall the solid (Cc) contained under the three first antecedents, be to that (Hh) contained under their three consequents; as the solid (Mm) contained under the three other antecedents, is to that (Qq) contained under the three remaining consequents.

L 4

Let

Let Hr and Qs be parallelepipeds on the bases HI and QR , of the same altitude with Cc and Mm respectively.

ϵ 21. 7. Then shall $Cc : Hr :: \text{base } CD : \text{base } HI^k$;

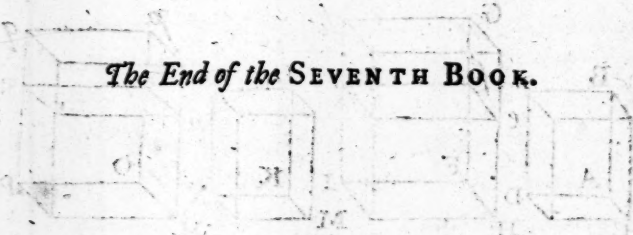
ϵ 11. 4. And $Mm : Qs :: \text{base } MN : \text{base } QR^k$. But the four bases, because of the proportionality of their sides, are themselves proportionals¹: and so, by equality, the ratio of Cc to Hr , is the same as the ratio of Mm to Qs . And the ratio of Hr to Hb is likewise the same as that of Qs to Qq (because $Hr : Hb :: Ff(AB) : FG^m :: KL(Oo) : Op^n :: Qs : Qq^m$.) Therefore shall the ratio of Cc to Hb , be also the same, as the ratio of Mm to Qq^o .

o 5. 4.

COROLLARY.

Hence the cubes of four proportional lines are proportional.

The End of the SEVENTH BOOK.



ELEMENTS

OF

GEOMETRY.

BOOK VIII.

POSTULATES.

1. **T**HAT, of any two unequal magnitudes, of the same kind, the less may be multiplied so often, till it exceed the greater.

2. That, a right-line may be taken so small, that the square thereof shall be less than any superficies assigned.

3. That, the circumference of a circle is greater than the perimeter (or the sum of all the sides) of any inscribed polygon; and less than the perimeter of any polygon described about the circle.

What is required to be granted, in the second of these three Postulates, might be effected and proved, in form, by means of the First; but being itself more obvious (if possible) than even that, it seemed unnecessary to make it depend thereon.

LEM.

L E M M A I.

If from the greater (AQ) of two unequal magnitudes (AQ, CD) there be taken the half (PQ), and from the remainder (AP) be again taken the half (PO), and so on, continually; there shall at length be left a magnitude, less than the least (CD) of the two magnitudes first propounded.

Take $DE = \frac{1}{2} CD$, $EF = \frac{1}{2} DE$, $FG = \frac{1}{2} FE$, &c.
and let this be so often done, till the multi-

- * Post. 1. ple CF exceed AQ^a. Let the proposed bisections of AQ, AP, &c, be continued till the parts PQ, OP, AO be equal in number to the parts EF, DE, CD. Now $AP (\frac{1}{2} AQ^b) \supset \frac{1}{2} CF^b \supset CE$. And, in the same manner, $AO (\frac{1}{2} AP) \supset \frac{1}{2} CE (CD)$; which was to be done.

S C H O L I U M.

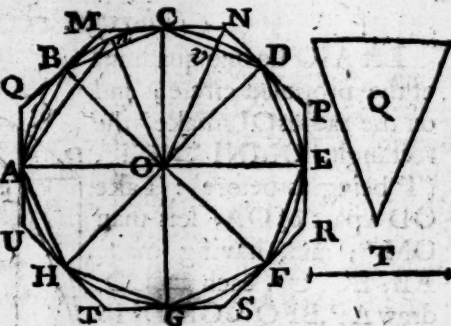
- When the magnitudes given (AQ, CD) are right-lines, a part, or measure (AS) of the one, less than the other, may be found at one operation^c; by taking AS the same part of AQ, as CD is of CF^c. For, the whole AQ being less than the whole CF^b, the part AS will also be less than the part CD^d.

T H E O R E M I.

Two polygons may be formed, the one in, the other about a given circle, which shall differ less from each other (and consequently from the circle itself) than by any assigned magnitude (Q) however small.

Let

Let T be the side of a square equal to, or less than Q ; and in the circle apply $An = T$: and, having drawn the two perpendicular diameters AE ,



Post. 2.

20. 5.

CG , proceed by a continual bisection of the angles at the center, till you arrive at an angle AOB less than the angle AOu subtended by An : Inscribe the regular polygon $ABCDEFGH$, by making the angles BOC , COD , &c. equal to AOB : and let another regular polygon $QMNPRTSU$, of the same number of sides, be described about the circle; which will exceed the inscribed one by a magnitude less than Q .

For, if to any angle N of the greater, ON be drawn, it will bisect the same, and will cut the side CD of the inscribed polygon at right-angles (in v): And so, the triangles OCN , OCN being equiangular, they (and consequently their doubles $OCND$, CND) will be in proportion to each other, as OC^2 to Cv^2 , or as AE^2 to AB^2 . And it is manifest, that the whole circumscribing polygon ($OCND + ODPE$, &c.) must be to its whole excess ($CND + DPE$, &c.) above the inscribed one, in the same proportion of AE^2 to AB^2 . But the first antecedent is less than the second, or than a square described about the circle: Therefore the first consequent ($CND + DPE$, &c.) is also $\propto AB^2$. $\propto An^2$ (T^2) $\propto Q$.

5. 5. and
Lem. 1.
Ax. 10.
and 4. 1.

30. 5.

12. 1.

19. 4.

24. 4.

1. 4.

Ax. 2.

2. 4.

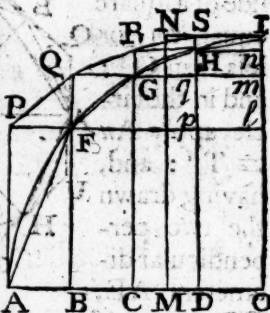
21. 1.

Hyp.

Other.

Otherwise.

Let AIO be one quadrant of the proposed circle; and on the radius OI, make the rectangle OMNI $= \frac{1}{4} T^2$, (T being as before): Take OD a part of OA, less than OM^a; and, having made AB, BC, &c. each = OD, draw AP, BFQ, CGR, DHS perpendicular^a to AO, meeting the circumference in A, F, G, H; through which points, parallel to AO^a, draw PF^l, QG^m, RHⁿ meeting AP, BFQ, CGR, in P, Q, R: Join PQ, QR, RS, SI, as also AF, FG, GH, HI. Then will the two polygons OAFGHI and OAPQRSI (whereof one is less, and the other greater than the quadrant) differ less from each other, than by $\frac{1}{4}$ of the proposed quantity Q.



For, that the former OAFGHI is less than the quadrant, in which it is inscribed, is manifest^a: And, that the latter is greater than the quadrant, will also plainly appear; seeing two sides AP, SI, only, touch the circumference^a; all the rest PQ, QR, RS, falling wholly above it, as being sides of triangles PFQ, QGR, RHS formed out of the circle^a. Now the excess of the polygon OAPQRSI above OAFGHI, is composed of the triangle PAF ($= \frac{1}{2} ODpl$) and of all the parallelograms PFGQ, QGHR^b, RHIS (for they are such^c, because PF (AB), QG (BC) are equal, as well as parallel^d). Therefore PFGQ being $= plmq$ ^e; QGHR $= qmnH$ ^e, &c. the said excess will consequently be $= \frac{1}{2} ODpl + lpSI$ ^b $\supset ODSI$ ^e $\supset OMNI$ ($\frac{1}{4} T^2$) $\supset \frac{1}{4} Q$ ^g; which was to be done.—This last construction is equally applicable to other curvilinear figures; the former is peculiar to the circle.

COROL-

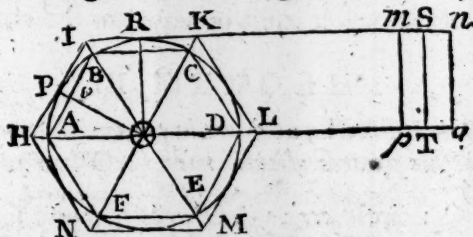
COROLLARY.

It follows from hence, that a magnitude, which is greater than any polygon that can be described in, and less than any polygon that can be formed about a given circle, must be equal to the circle itself: seeing that a polygon may be inscribed, which (as well as that formed about the circle) shall exceed any quantity less than the circle itself, be the difference ever so small; and because a polygon may be formed about the circle, which (as well as that in the circle) shall be less than any quantity that exceeds the circle.

THEOREM II.

Every circle (ACE) is equal to a rectangle (ORST) under the radius thereof (OR) and a right-line (OT) equal to half the circumference.

It is evident, in the first place, that the proposed rectangle ORST is greater than any poly-



gon ABCDEF that can be described in the circle: For, drawing OA, OB, &c. And also Ov perpendicular to AB; it is plain, that the triangle AOB (^h $Or \times \frac{1}{2} AB$) will be less than ⁱ $OA \times \frac{1}{2} AB$ ^h Cor. to 2. 2. (or $OR \times \frac{1}{2} AB$): And* in the same manner, BOC ⁱ $\square OR \times \frac{1}{2} BC$, &c. Consequently, the whole polygon ABCDEF is less than ^k $OR \times \frac{1}{2} AB + OR \times \frac{1}{2} BC$, &c. that is ^l, less than a rectangle (Om) ^k Ax. 2.1. under OR and Op = half the perimeter (AB + BC + CD, &c.) ^l 5. 2. But this rectangle (Om) is, itself, less than

than OS, because, Op (half the perimeter of the polygon) is less than OT^m (half the circumference of the circle). Consequently the polygon ABCDEF is less than the rectangle OS.

But, secondly, it will appear that the same rectangle ORST is less than any polygon HIKLMN that can be described about the circle: For, if OH, OI, &c. be joined, and the radius OP be drawn to the point where HI touches the circle; then will the triangle HOI = ⁿOP × $\frac{1}{2}$ HI (= OR × $\frac{1}{2}$ HI). In the very same manner IOK = OR × $\frac{1}{2}$ IK, &c. and therefore the whole polygon HIKLMN = ^oOR × $\frac{1}{2}$ HI + OR × $\frac{1}{2}$ IK, &c. = ^pa rectangle (Or) under OR and Oq = half the perimeter (HI + IK + KL, &c.); which rectangle is, manifestly greater than OS, since Oq (= half the perimeter of the polygon) is greater than OT^q.

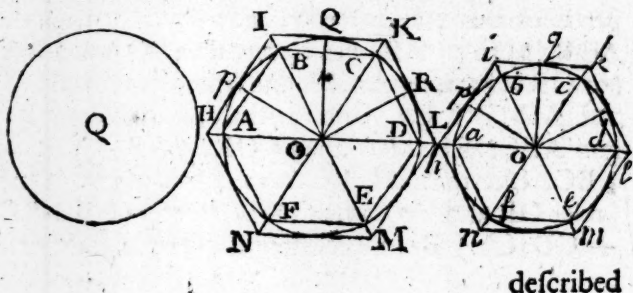
Seeing therefore, that the rectangle OS is greater than any polygon that can be described in the circle, and less than any polygon that can be described about the circle; it must be equal to the circle^r.

^r Cor. to
1. 8.

THEOREM III.

All circles (ACE, ace) are in proportion to one another, as the squares of their radii (AO², ao²).

Let Q: circle ace :: AO²: ao²; then I say, that Q = circle ACE. For, first, it is evident that Q is greater than any polygon ABCDEF that can be



described in the circle ACE : Because, if another polygon *abcdef*, similar thereto, be^a described in the circle *ace*; then will polyg. ABCDEF : polyg. *abcdef* (:: 'AO^a : ao^a) :: Q : " circle *ace*; where the first consequent (polyg. *abcdef*) being less than the second (or, than the circle in which it is inscribed^b) it is manifest, that the first antecedent ABCDEF must also be less than the second Q^c.

In the same manner it will appear, that Q is less than any polygon HIKLMN that can possibly be described about the circle ACE : For, if about the other circle *ace*, a similar polygon *biklmn* be described^d; then will HIKLMN : *biklmn* (:: 'AO^a : ao^a) :: Q : " circle *ace*; where the first consequent (*biklmn*) being greater than the second (*ace*^b), the first antecedent HIKLMN must therefore be also greater than the second Q.

Therefore, seeing that Q is greater than any polygon that can be described in the circle ACE, and less than any polygon that can be described about the circle; it must be equal to the circle^d.

SCHOLIUM.

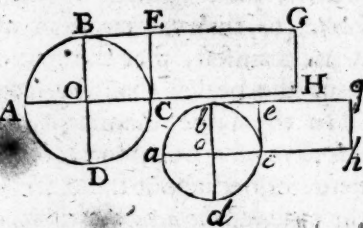
After the same manner, other similar curvilinear figures are proved to be in proportion, as the squares of their diameters, or other homologous dimensions; by means of the second construction of the first proposition; it being very easy to demonstrate, that the polygons formed from thence, whether both within, or both without two similar figures, will themselves be similar.

THEO-

THEOREM IV.

The circumferences of all circles (ABCD, abcd) are in the same proportion as their radii (OB, ob).

Let OE, oe be squares on the radii OB, ob; and let OG, og be two rectangles contained under the same radii and right-



semi-circumferences ABC, abc. Then, these rectangles being equal to the circles themselves^d, it will therefore be, OE : OG :: oe : og^e. And in this same ratio are^f also the bases OC, OH; oc, ob : whence (by equality and alternation) OC (OB) : oc (ob) :: OH : ob :: 2OH (circumf. ABCD^g) : 2ob (circumf. abcd).

^d 2. 8.

^e 3. 8.

^f 7. 4.

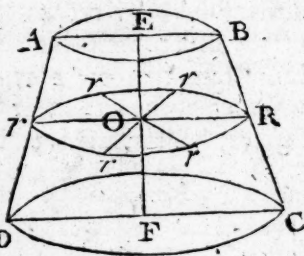
^g Hyp.

LEMMA 2.

If a solid (AC) generated by the revolution of any plane figure (EBCF) about a quiescent axis (EF), be cut by a plane perpendicular to the axis; the section will be a circle, having its center in the point (O) where it meets the axis.

For, from O, in the generating plane EBCF, draw OR perpendicular to the axis EF, meeting BC in R.

Then, since this line OR, during the whole revolution, every where



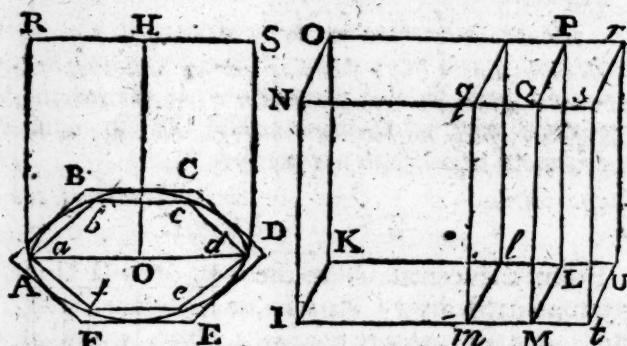
larity

THEOREM V.

Every cylinder is equal to a rectangular parallelepipedon of equal base and altitude.

I say, if the base *ace* of the cylinder *aS* be equal to the base *IKLM* of the rectangular parallelepipedon *IP*, and the altitude *OH* of the former be also equal to the altitude *KO* of the latter; then the two solids will be equal.

For, first, it is evident, that the cylinder exceeds any parallelepipedon (*Ip*), of the same given altitude, whose base *Ikln* is less than the base (*ace*) of



- the cylinder. Because a polygon (*abcdef*) may be described in the circle *ace*, that shall exceed *IKln*¹; upon which, an upright prism (of the given altitude) may be constituted²; which will be less than the cylinder as being wholly contained therein; since (by *Lemma 3.*) all right-lines drawn perpendicular to the base, in the planes of the sides³, from any points in *ab*, *bc* &c. fall wholly within the cylinder, and consequently the planes themselves, in which they are. But this contained prism is greater than the parallelepipedon *Ip*⁴: therefore the cylinder itself must, necessarily, be greater than *Ip*⁵.

In

In like manner it will appear, that the cylinder is less than any parallelepipedon Ir (of the same altitude) whose base $IKvt$ exceeds that of the cylinder: For a polygon ($ABCDEF$) may be described about the circle *see* that shall be less than $IKvt$; upon which a prism may be constituted ^{1. 8.} v , which, tho' less than Ir ; will, nevertheless, exceed the cylinder ^{18. 7.} v . ^{21. 7.} x . ^{7. Ax. 2.} 7 $Ax. 2.$

Therefore, seeing that the cylinder can neither be less, nor greater than IP ; it must necessarily be equal to it.

COROLLARY.

Hence, whatever is demonstrated in the 21st, 22d, and 23d Theorems of the preceding Book, with respect to the proportions of prisms, holds equally true in cylinders also; being equal to prisms of equal base and altitude ^{20. 7.} * .

^{20. 7.} *
and 5. 8.

SCHOLIUM.

From the same demonstration, it will likewise appear, that every regular solid, whose sections, by planes perpendicular to the base, are all rectangles, is equal to a parallelepipedon of equal base and altitude; and consequently, that all solids of this kind (which may be comprehended under the name of *Cylinderoids*) will be equal among themselves, when their altitudes, as well as bases, are equal.

LEMMA 4.

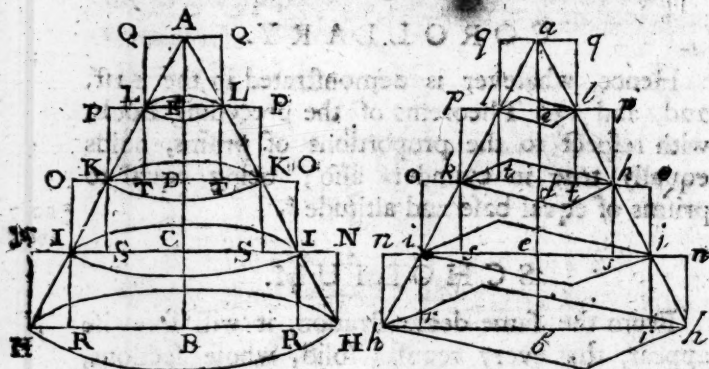
If two solids (HAH , hah) of the same altitude, have their sections by planes parallel to the bases, at all equal distances therefrom, equal to each other; it is proposed to demonstrate (under certain restrictions specified hereafter) that the solids themselves will be equal.

M 2

Let

Let $II, KK \&c. ii, kk, \&c.$ be sections of the two solids by planes parallel to the bases HH, bb , dividing the altitudes AB, ab into parts $BC, CD \&c. bc, cd \&c.$ all mutually equal to each other. Then, every two corresponding sections being equal^m ($HH = bb, II = ii, \&c.$) the upright solids $HNNH, bmbb; IOOI, iooi \&c.$ formed thereon, will also be respectively, equal one to anotherⁿ, whether they be prisms, cylinders, or cylinderoids, that is, whether the sections themselves be right-lined figures, circles, or curvilinear figures of any other kind.

20. 7.
and Sch.
5. 8.



Now if these sections $HH, II \&c.$ be supposed to decrease, from the base upwards, so that the solids ($HNNH, IOOI \&c.$) formed upon them, may exceed the correspondent parts ($IIII, IKKI \&c.$) of the given solid HAH ; it is manifest, that the sum of all the said solids ($HNNH + IOOI \&c.$) will likewise exceed the whole proposed solid HAH ^o. But, if within HAH , on the same sections (but on contrary sides thereof) another series of such solids $IRRI, KSSK \&c.$ be formed; the sum of all these will, manifestly, be less than the proposed solid HAH , in which they are contained^p. And it is also evident, that this last series will be less than the former ($HNNH + IOOI \&c.$) by the greatest of these

these solids HNNH; because (this one, alone, being excepted) to every other solid of the rank, an equal, in the contained rank, may be assigned, and *vice versa*: For $IRRI = IOOI$, $KSSK = KPPK$, $LTTT = LQQL$. 20. 7.
and Sch.
5. 8.

Now, since the altitude BC, of the solid HNNH, whereby the contained, and containing series, differ from each other, may be taken so small a part of BA, that the solid itself shall be less than any assigned magnitude whatever; it is manifest (from the reasoning in Corol. to Theor. I.) that a magnitude, which is greater than any series of solids (of the kind above specified) that can be formed within the proposed solid HAH, and less than any series that can be formed about HAH, must be equal to HAH. But the solid *bab*, being greater than any series of solids (*irri + kssk* &c.) contained therein, is therefore greater than any series of solids $IRRI + KSSK$ &c. contained in HAH (*these*, being, respectively, equal to *those*): And the same solid *bab*, being less than any series of solids (*bnnb + iooi* &c.) formed about it, is also less than any series of solids ($HNNH + IOOI$ &c.) that can be formed about HAH. Therefore the solid *bab* is equal to HAH. Lem. 1.
and 22. 7.
Ax. 2.

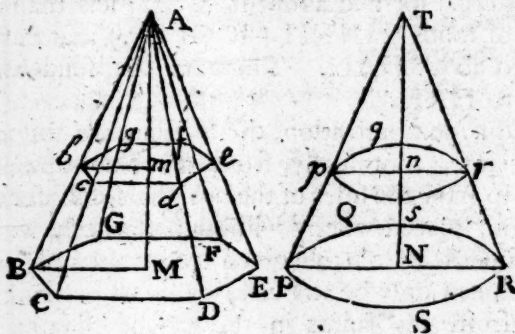
In this demonstration, the sections are supposed to decrease, continually, from the bases upwards; so as to have the sides of the upright solids formed thereon, placed wholly without, or wholly within, the superficies of the given solids HAH, *bab*: Which can only be the case, when all perpendiculars, from any points in the surface of either, to the plane of the base, fall within the limits of the base. If, however, the sections be supposed to decrease to a certain distance, only, and then to increase again, the two solids will, still, appear to be equal: Because the parts of the one, terminated

by such limits of decrease, or increase, will (by the same demonstration) be respectively equal to the correspondent part of the other. But, as no such solids have a place in the *Elements of Geometry*, to say more about them here, would be improper.

LEMMA 5.

If pyramids and cones (ABCDEFG, TPQRS) having equal altitudes (AM, TN), be cut by planes parallel to the bases; the sections (bcdefg, pqrs) at all equal altitudes (Mm, Nn), will be in the same proportion as the bases.

For the plane *bcdefg* being parallel to BCDEFG, thence is *bc* parallel to BC^a, *bg* to BG, &c, and consequently the angle *cbg* = CBG^b, *bcd* = BCD, &c. also *bc* : BC (∴ *Ab* : AB^c) ∴ *bg* : BG. And, in the same manner, the sides about the other equal angles are proportional. Therefore, the two po-



^a Def. 14.

^b 4.

^c 26. 4.

^d Cor. to

11. 4.

^e Cor. to

7. 7.

^f 4. 1.

lygons *bcdefg*, BCDEFG being similar^d, they are in proportion^e, as *bc*² to BC², or as *Ab*² to AB², or, lastly, as AM² to AM²; because (BM and *bm* being drawn) the angles AMB, *amb* will be right-ones^f, and *bm*, therefore, parallel to BM^h.

But

But the section *pqr* is *also* to the base PQRS in the same proportion of Am^2 (Tn^2) to AM^2 (TN^2); because *pqr*, PQRS, being circles, they are as the squares of their radii pm , PN^2 , and consequently as Tn^2 to TN^2 . Therefore, seeing that the two sections have both the same ratio to their respective bases, the proposition is manifest.
Lem. 2.
3. 8.
Cor. to
11. 4.
2. 4.

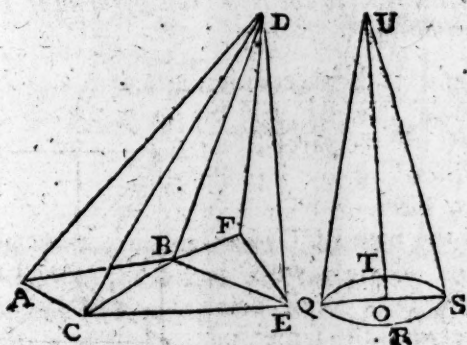
COROLLARY.

It appears from hence, that the section of any pyramid, by a plane parallel to the base, is similar to the base.

THEOREM VI.

All pyramids of the same altitude standing upon equal triangular bases, are equal among themselves; and every such pyramid (ABCD) is equal to a cone (QRSTU) of equal base and altitude.

CASE I. If the perpendicular, let fall from the vertex D of the pyramid upon the plane of the base ABC, falls not out of the base, or beyond the limits of the triangle: Then it is manifest, from Lemma 4, seeing the sections of the solids ABCD,



M 4

QRSTU,

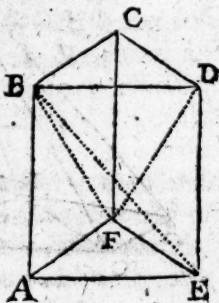
QRSTU, at all equal distances from the bases, will
 Lem. 5. be equal; that the solids themselves will likewise
 be equal.

CASE II. If the perpendicular (DE) from the
 vertex to the plane of the base, falls beyond the li-
 mits of the triangle: Then, to the point E where
 it meets the plane, let BE and CE be drawn; and
 on BE let a triangle EBF be described equal to
 ABC (or QRST), and let F, D be joined. So
 shall the pyramid CBFED, standing on the base
 CBFE, be equal to the pyramid CABED, stand-
 ing on the equal base CABE; from each of
 which, let the common pyramid CBED be taken
 away; and there will then remain the pyramid
 Ax. 5. BFED = pyramid ABCD: But the former of
 these is (by case 1.) equal to the cone QRSTU;
 therefore it is evident, that the latter ABCD will
 Ax. 1. also be equal to the cone QRSTU; and, conse-
 quently, that all pyramids of the same altitude,
 standing on equal triangular bases, will be equal
 among themselves; seeing every such pyramid is
 equal to a cone (QRSTU) of equal base and al-
 titude.

THEOREM VII.

Every prism (ABCDEF) having a triangular
 base (AFE) is equal to the triple of a pyramid of the
 same base and altitude.

In the planes of the three sides,
 let the diagonals BE, BF, FD be
 drawn. Then will the part FBCD
 of the prism cut off by a plane
 extended by FB and FD, be a
 pyramid on the base BCD, hav-
 ing the same altitude with the
 prism itself, both solids being
 contained between the same pa-
 rallel planes AFE, BCD. More-



over;

over, the remaining part $FABDE$ of the prism, if a plane be extended by FB and FE , will be divided into the two pyramids $FBAE$, $FDDE$, which are equal to each other^a, as standing on the equal^b triangular bases ABE , BDE . But the former of these pyramids $FBAE$, if B be now considered as the vertex thereof, will appear, also, to be equal to the first mentioned pyramid $FBCD$,^a the two bases AFE , BCD (as well as the altitudes) being equal^b. Therefore, since the three triangular pyramids ($FBCD$, $FABE$, $FBDE$) into which the prism is resolved, are all equal to each other, the proposition is manifest.

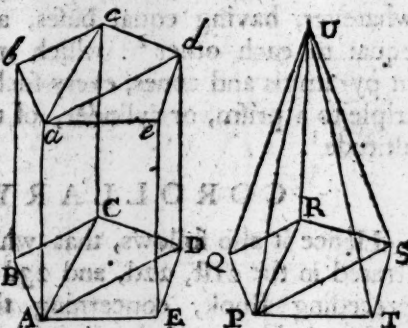
COROLLARY.

Hence, every prism having a triangular base, is equal to the triple of any pyramid of the same altitude, standing upon an equal triangular base^a. ^{c 6. 8. and Ax. 1.}

THEOREM VIII.

If a prism ($AbcE$) and a pyramid ($PQRSTU$) stand upon equal and similar bases ($ABCDE$, $PQRST$) and have both the same altitude; the prism will be equal to the triple of the pyramid.

If the bases be resolved into triangles, ABC , ACD &c. it is manifest, that $BbacCA$ will be a prism, on the base ABC ; because Cc being equal Bb and parallel to



Aa , $AacC$ will be a parallelogram^a (as well as $BbaA$ and $BbcC$).

^a Def. 6. and 8. 7.
^b 26. 1.
There-^c Def. 6. 7.

Therefore $BbacCA$ is equal to the triple of the pyramid $PQRU$ ^s, standing on an equal ^a base (PQR) . And, in the same manner, the prism $AacdDC$, on the base ACD , is equal to the triple of the pyramid $PRSU$, on the equal base PRS ; and so on. Therefore, also, shall the whole prism Ad , on the base $ABCDE$, be equal to the triple of the whole pyramid $PQRSTU$, on the equal base $PQRST$.

^s Cor. to
7. 8.
^a Hyp and
Ax. 10.

COROLLARY I.

Hence, all pyramids having the same base and altitude, are equal; being like parts of one and the same prism.

COROLLARY II.

Hence, also, all prisms having the same base and altitude, are equal; being equimultiples of one and the same pyramid.

COROLLARY III.

Therefore it appears, that every prism inclining on its base, as well as every upright one, is equal to a rectangular parallelepipedon of equal base and altitude^s; and, consequently, that all prisms ^s 20. 7. and Ax. 1. whatever, having equal bases, and altitudes, are equal to each other^s: which must be also true in pyramids and cones, every such solid being sub-triple to a prism, or cylinder, of the same base and altitude^s. ^s 8. 8. and 6. 8.

COROLLARY IV.

Hence it also follows, that whatever is demonstrated in the 21st, 22d, and 23d Theorems of the preceding Book, concerning the proportion of prisms, holds equally in pyramids and cones; *these* being like parts of *those*^s.

COROL.

COROLLARY V.

It follows, moreover, that all corresponding frustums of pyramids and cones of the same altitude, are also, in proportion, as their bases. For, the sections, at all equal altitudes, being in that proportion^m, the parts cut off (as well the wholes) will be in the same proportionⁿ; and, consequently, the remaining parts likewise^o. Lem. 5.
Cor. 4.
3. 4.

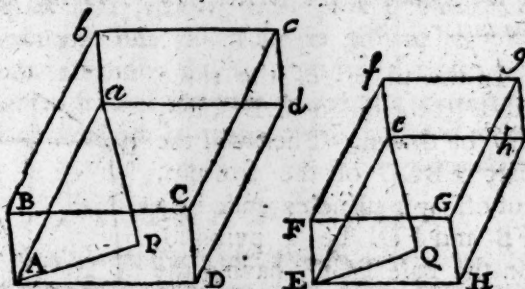
COROLLARY VI.

Lastly, it will appear, that all cones, which have their altitudes and the diameters of their bases directly proportional, are in the triplicate ratio of their altitudes^p; being to each other in the same^q proportion with prisms of equal base and altitude, whereof they are like parts^r. 24. 7.
Cor. 3.
and Cor.
to 1. 4.

THEOREM IX.

All similar prisms, and pyramids, are in the triplicate ratio of their altitudes.

From the extremes of the homologous sides Aa , Ee , upon the bases $ABCD$, $EFGH$ of the proposed solids Ac , Eg , let fall the perpendiculars



aP , eQ . The angle BAD being $= FEH$, $BAA^p = FEe^p$ Def. 5. 7.
 $= FEE$, and $DAa = HEE$, thence is $aP : eQ :: A^p : e^p$ Def. 14.
 $aA : eE :: AB : EF :: AD : EH$. Therefore $aA^2 : eE^2 :: AB^2 : EF^2 :: AD^2 : EH^2$.
4.
two

- two upright prisms constituted on the bases ABCD, EFGH, of the same altitudes (aP , eQ) with the two solids Ac , Eg , will be similar, the one to the other^r; and, therefore, in the triplicate ratio of the altitudes^s. But the solids Ac , Eg , when taken as prisms, are respectively equal to the said upright ones; and, when taken as pyramids, are like parts of them^t. Therefore the solids Ac , Eg are also in the triplicate ratio of the altitudes aP and eQ ^u.
- ^r Def. 5. 7.
^s 24. 7.
^t Cor. 3.
^u to 8. 8.
^v Cor. 1. 4.

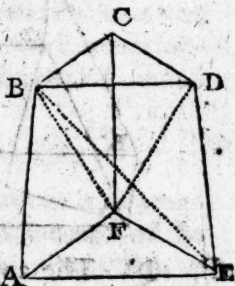
COROLLARY.

- Because $aP : eQ :: Aa : Ee :: AB : EF$ &c. it follows, that all similar prisms, and pyramids, are to one another, in the triplicate ratio of the homologous sides of the like planes by which they are bounded^v.
- ^v Cor. 1.
 to 5. 4.

THEOREM X.

The frustum (ABCDEFA) of any pyramid having a triangular base, is equal to a whole pyramid, of the same base and altitude, together with two other pyramids, that are in proportion thereto; the one, as any side (BD) of the upper base (BCD) is to its correspondent (AE) of the lower base (AFE); and the other, as the square of the former side is to the square of the latter.

- In the planes of the three sides, let the diagonals BE, BF, FD be drawn. Then will the part FBCD of the frustum, cut off by a plane extended by FB and FD, be a pyramid, on the base BCD, having the same altitude with the frustum itself^z, both solids being contained between the same^a parallel planes AFE, BCD.
- ^z Def. 16.
^a Def. 15.
 7.



More-

Moreover, the remaining part FABDE of the frustum, if a plane be extended by FB and FE, will be divided into the two pyramids FBDE, FAFE, having the same ratio, one to the other, as their bases BED, ABE^b, or as BD to AE^c. But the latter of these pyramids (BAFE), taking B as the vertex thereof, has the same base and altitude with the frustum given: And the pyramid FBCE (first mentioned) is therefore, in proportion thereto, as the base BCD to the base AFE^d, that is (because the bases are similar^e), as BD² to AE²: whence the proposition is manifest.

^b 21. 7.
and Cor.
4. to 8. 8.
^c 7. 4.
^d 21. 7.
and Cor.
4. to 8. 8.
^e Cor. to
Lem. 5.
^f 26. 4.

COROLLARY.

Since, of the three solids (FAFE, FBDE, FBCE) into which the proposed frustum is divided, the ratio of the first and third, is the duplicate of that of AE to BD, or of the ratio of the first to the second^g; it is evident, that these three solids are proportionals^h. From whence it appears, that the frustum of any triangular pyramid is equal to two (whole) pyramids of the same altitude, on bases equal to the two opposite bases of the frustum, and to a third pyramid, which is a mean proportional between the two former. And it is also evident, that whatever is above demonstrated, in relation to triangular pyramids, must hold equally in all pyramids and cones, whatever: Because every such solid is equal to a triangular pyramid, of equal base and altitudeⁱ; and every frustum of the one, also equal to the corresponding frustum of the other^k.

^g Cor. to
27. 4.
^h Def. 7. 4.

ⁱ Cor. 3.
to 8. 8.
^k Cor. 5.
to 8. 8.

LEMMA 6.

If with radii, respectively equal to the three sides of any right-angled triangle, three circles be described; that whose radius is equal to the hypotenuse, will be equal to both the other two, taken together.

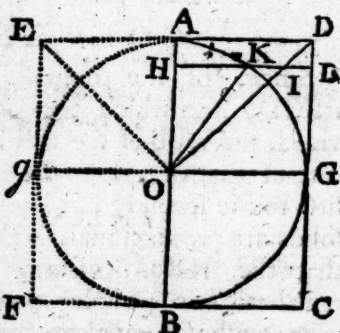
It

It has been proved (*in Theor. 3.*) that circles are in proportion, as the squares of their radii; therefore the demonstration here, is the very same, as in similar right-lined figures (*Theor. 29. Book 4.*): Which (if necessary) you may consult.

THEOREM XI.

Every sphere is two-thirds of its circumscribing cylinder (Or, of a cylinder of equal diameter and altitude.)

Let AB be the axis about which the sphere and cylinder are generated, by the revolution of the semi-circle AGB and the rectangle ADCB¹; let HL be any right-line perpendicular to AB, meeting DC in L, and the periphery of the semi-circle in K; and from the center O, let OK and OD, intersecting HL in I, be drawn.



¹ Def. 12.
and 14.

² Hyp. Since AO is = AD², and HI parallel to AD², therefore is HI = OH²: But OHK being right-angled at H, the circle whose radius is OH (or HI) will (*by the preceding Lem. and Ax. 5.*) be equal to the difference of the two circles whose radii are OK (HL) and HK³: Or, in other words, the circle⁴ described by HI, or the section of the cone generated by the triangle AOD, in its revolution about the axis AB, will be equal to the difference of the two circles generated by HL and HK; that is, equal to the annulus described by KL⁵, or the section of the solid which remains, when the sphere is taken out of the cylinder. Therefore, seeing these

² Hyp.

³ 4. 1.

⁴ 14. 4.

⁵ Lem. 6.

⁶ Ax. 2. 1.

these two sections are, every-where, equal to each other, the solids themselves will likewise be equal; Lem. 4.
 that is, the cone (EOD) will be equal to the excess of the cylinder (GDEg) above the inscribed hemisphere (GAg): whence, as the cone, or excess, is one-third part, of the cylinder; the hemisphere must necessarily be equal to the two remaining thirds. And what is here proved, with respect to the halves of the proposed solids, holds equally in the wholes. Therefore every sphere is two-thirds of its circumscribing cylinder. 6. 8. and Cor. 3. to 8. 8.

COROLLARY I.

Hence, a cone, hemisphere and cylinder, of the same altitude, and standing upon equal bases, are in proportion, as the numbers 1, 2 and 3, respectively.

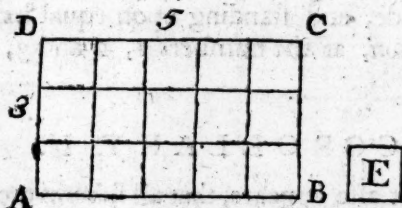
COROLLARY II.

Hence it also appears, that all spheres are to each other in the triplicate ratio of their diameters; being in the same proportion as the circumscribing cylinders, whereof they are like parts. Cor. to 5. 8. and 24. 7.

END of the ELEMENTS.

OF THE MENSURATION

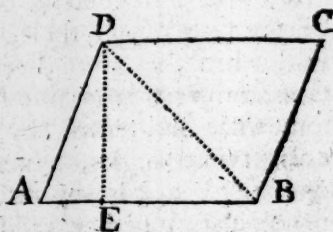
OF Superficies and Solids.



EVERY quantity is measured by some other quantity of the same kind; as a line by a line, a surface by a surface, and a solid by a solid: And the number which shews how often the lesser, called *the measuring unit*, is contained in the greater, or quantity measured, is called *the content of the quantity so measured*. Thus, if the quantity to be measured be the rectangle ABCD, and the little square E, whose side is one inch, be the measuring unit propounded; then, as often as the said little square is contained in the rectangle, so many square inches the rectangle is said to contain: So that, if the length DC be supposed 5 inches, and the breadth AD 3 inches; the content of the rectangle

rectangle will be 3 times 5, or 15 square inches: Because, if lines be drawn parallel to the sides, at an inch distance one from another, they will divide the whole rectangle ABCD into 3 times 5, or 15 equal parts, of one inch each. And, generally, whatever the measures of the two sides may be, it is evident (*from El. 7. of 4.*) that the rectangle will contain the square E, as many times as the base AB contains the base of the square, repeated as often as the altitude AD contains the altitude of the square. Therefore, *to find the content of any rectangle, multiply the base by the altitude, and the product will be the answer.* Thus, let the length be 18 inches, and the breadth 15; then the content will be 15 times 18, or 270 square inches.

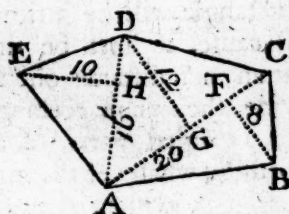
The method of finding the content of a rectangle being thus known, the content of any parallelogram ABCD, or triangle ABD, will also be known; the former of



these figures being equal to a rectangle of the same base and altitude; and the latter equal to the half of such a rectangle (by Cor. 2. to 2, 2.). Therefore, *multiply the base by the perpendicular, for the content of any parallelogram; and the base by half the perpendicular, for that of any triangle.* Thus, for example, let the base AB be 18 feet, and the perpendicular DE 12 feet; then the content of the parallelogram will be 216, and that of the triangle 108, square feet.

Of the Mensuration of

From the manner of finding the area of a triangle, the area of any right-lined plain figure, as ABCDE, may be determined, by dividing the whole into triangles, and finding the content of each triangle. Thus, let the dividing lines AC and AD, be 20 and 16 inches, and the perpendiculars BF, DG, EH, falling thereon, 8, 12, and 10, respectively; then, the content of the



Triangle $\left\{ \begin{array}{l} ABC \\ ACD \\ ADE \end{array} \right\}$ being $\left\{ \begin{array}{l} 80 \\ 120 \\ 80 \end{array} \right\}$, it is evi-

dent, that the content of the whole figure will be the sum of all these, or 280 square inches. But, when the given lines are expressed by fractions, or very large numbers, the work will be somewhat shortened, by finding the content of every two triangles, having the same base, at one operation; that is, by first adding the two perpendiculars together, and then multiplying half their sum by the common base of the two triangles. Thus, in the last example, the half-sum of the two perpendiculars BF and DG being 10, if this number be, therefore, multiplied by 20 the measure of the common base AC, the product, which is 200, will be the content of the trapezium ABCDA; to which 80, the content of the triangle ADE, being added; the sum will be 280, the same as before. But, if the polygon proposed be a regular one, that is, one whose sides, and angles are all equal, the shortest way

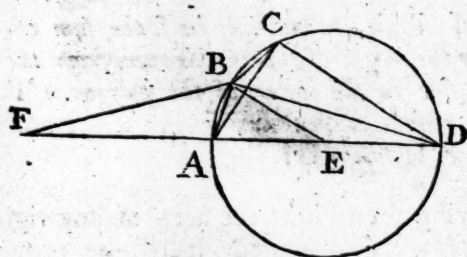
way of all, is, to multiply half the sum of all the sides by the length of the line drawn from the middle of any side to the center of the polygon. The reason of which is obvious, from the demonstration to Theor. II. B. VIII.

Having shewn how the area of any right-lined figure may be computed, it will be proper here, to say something with regard to the area, and periphery of the circle.

It is well known, that to determine the true area of a circle, and to find a right-line exactly equal to the circumference thereof, are looked upon, by mathematicians, as absolutely impossible: But, though neither the one nor the other can be accurately known, yet several Ways have been invented by which they may be approximated, to any assigned degree of exactness. That, which I am now going to lay down, though less expeditious than some others, seems, nevertheless, to be the most proper for this place, as depending on the most simple and evident principles: I shall therefore begin with premising the following

L E M M A.

If AD be a diameter, and AB, BC two equal arcs of the same circle, and if the chords DB, DC be drawn; then, I say, that $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$.



For, if in DA produced, there be taken $AF = DC$, and BF, BA, BC and the radius BE be drawn; then, the external angle FAB, of the trapezium ABCD, being equal to the internal opposite angle DCB (*by* 17. 3.) also $AF = DC$, and $AB = CB$ (*by* Hyp.); it is evident, that FB is also $= DB$, and consequently the angle $F = FDB = DBE$: And so the isosceles triangles DEB, DBF being equiangular, it will be as $DE (\frac{1}{2}AD)$: $DB :: DB : DF (DC + AD)$; and consequently $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$. *Q. E. D.*

COROLLARY.

Hence, if the diameter AD be denoted by the number 2, the chord DB will be denoted by $\sqrt{DC + 2}$: whence, it appears, that, if the measure of the supplemental-chord of any arch be increased by the number 2; the square-root of the sum will be the supplemental-chord of half that arch.

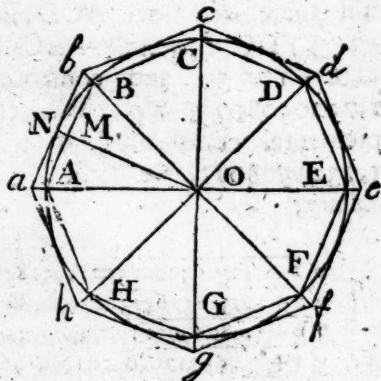
Now, to apply this to the matter proposed, that is, to the finding of the area and circumference of the circle; let the arch ABC be taken equal to $\frac{1}{3}$ of the semi-periphery ACD; then will the chord AC be equal to the radius AE (*by* 29. 5.); and

and therefore, since ACD is a right-angle (by 13. 3.) $DC^2 (= AD^2 - AC^2, \text{ by } 8. 2.)$ will be $= 4 - 1 = 3$; and consequently $DC = \sqrt{3} = 1,7320508075$, &c. Wherefore, seeing the supplemental chord of $\frac{1}{4}$ of the semi periphery is 1,7320508075, we shall by the preceding Corollary,

$$\left. \begin{array}{l} \sqrt{2 + 1,7320508075} = 1,9318516525 \\ \sqrt{2 + 1,9318516525} = 1,9828897227 \\ \sqrt{2 + 1,9828897227} = 1,9957178465 \\ \sqrt{2 + 1,9957178465} = 1,9989291743 \\ \sqrt{2 + 1,9989291743} = 1,9997322757 \\ \sqrt{2 + 1,9997322757} = 1,9999330678 \\ \sqrt{2 + 1,9999330678} = \sqrt{3,9999330678} \end{array} \right\} \begin{array}{l} \text{for the supplemental} \\ \text{chord of} \end{array} \left\{ \begin{array}{l} \frac{1}{8} \\ \frac{1}{4} \\ \frac{3}{8} \\ \frac{1}{2} \\ \frac{5}{8} \\ \frac{3}{4} \\ \frac{7}{8} \\ \frac{1}{1} \end{array} \right\} \begin{array}{l} \text{of the semi-periphery.} \end{array}$$

Now, therefore, since it is found that 3,9999330678 is the square of the supplemental-chord of $\frac{1}{8}$ of the semi-periphery, let this number be subtracted from 4 the square of the diameter, and the remainder 0,0000669322 will be the square of the chord of the same arch; therefore the chord itself being $= \sqrt{0,0000669322} = 0,00818121$, let this number be multiplied by 768, or twice 384, and the product 6,28317 will be the perimeter of a regular polygon of 768 sides, inscribed in the circle; which, as the sides of the polygon very nearly coincide with the circumference of the circle, must also express the length of the circumference itself, very nearly.

But, in order to shew how near this is to the truth, let AB represent one side of a regular polygon of 768 sides, inscribed in the circle (whose length, we have found above, to be 0,00818121) and let *ab* be a side of another similar polygon, described



about the circle; and from the center O let ON be drawn, bisecting AB and *ab* in M and N: Then, since $AM = \frac{1}{2}AB = 0,0040906$, and $AO = 1$, it is plain that $OM^2 (AO^2 - AM^2)$ will be $= 0,99998327$, and consequently $OM = 0,99999163$; whence, because of the similar triangles AOB, *aOb*, &c. we have $0,99999163 (OM) : 1 (ON) :: AB : ab :: 6,28317$ (the perimeter of the inscribed polygon) : $6,28322$ the perimeter of the circumscribed polygon. But the circumference of the circle being greater than the perimeter of the inscribed polygon, and less than that of the circumscribed one, it must, consequently, be greater than $6,28317$, and less than $6,28322$; and must, therefore, be equal to $6,2832$, very near; since this number exceeds the perimeter of the inscribed polygon by no more than $0,00003$, and is less than the perimeter of the circumscribed one by $0,00002$, only.

From the periphery thus found, the area of the circle will also be known; being equal to the product of half the periphery into the radius (by 2, 8.) that is, $= 3,1416 \times 1 = 3,1416$.

There-

Therefore, since it is proved (in *Theor.* 3 and 4. of 8.) that the peripheries of circles are in proportion as their diameters, and the circles themselves as the squares of those diameters; it follows, that, as 2 is to 6,2832, or as 1 to 3,1416 :: the diameter of any circle to its periphery; and as 4 to 3,1416, or as 1 to 0,7854 :: the square of the diameter to the area.

But, if you had rather have the proportions in whole numbers, and the case proposed does not require any great degree of accuracy; then, instead of the foregoing, those of *Archimedes* may be used, viz. 7 : 22 :: diam. : circumf. and 14 : 11 :: square diam. : area. Which proportions differ but little from those above, as will appear from the following example: wherein the diameter of a circle being given 28, its circumference and area are required. Here, according to the first proportions, I multiply 28 by 3,1416 for the circumference, and the square of 28 (or 784) by 0,7854 for the area; and there results 87,964 and 615,75, respectively. But, according to the proportions of *Archimedes*, the circumference will be found equal to 88, and the area 616; which differ very little from the former.

By knowing the proportion between the diameter of a circle and the circumference, and between the square of the diameter and the area, the convex superficies of solid bodies may be determined. Thus,

The convex superficies of a cylinder is found, by first finding the circumference of the base, and then multiplying by the altitude of the solid. Therefore, if to that product, the area of the two circular ends be added, the sum will be the whole superficies of the cylinder.

Of the Mensuration of

To find the convex superficies of a cone, multiply half the length of the slant side thereof by the circumference of the base.

The convex superficies of any frustum of this solid is found, by multiplying the sum of the peripheries of the two ends into half the length of the slant side of the frustum.

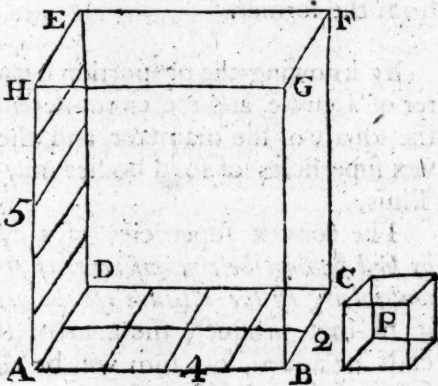
To find the superficies of a sphere, multiply the periphery of the greatest, or generating, circle by its diameter: Or, multiply the square of the diameter by 3,1416.

The convex superficies of any segment of a sphere is found, by multiplying the periphery of the greatest circle of the sphere into the altitude of the segment.

The demonstration of these last rules, for finding the curve surfaces of solid bodies (which is not given in the *Elements*, for reasons mentioned hereafter) is inserted at the end of this section.

OF THE MENSURATION OF SOLIDS.

As every superficies is measured by a square, whose side is unity (as one inch, one foot, one yard, &c.) so every solid is measured by a cube whereof the side is also an unit. Thus,



let

let the solid to be measured, be the rectangular parallelepipedon AF, and let the cube P, whose side is one inch, be the *measuring unit*; also let the length AB, of the base AC, be 4 inches, the breadth BC 2 inches, and the altitude AH of the solid 5 inches: Then, because the area of the base ABCD is 2 times 4 (or 8) square inches, it is easy to conceive, that, if the solid were to be only one inch high (instead of 5), the content thereof would be just the same number (8) of cubical inches; because then, upon the eight equal squares into which the whole base ABCD is divisible, a cube of one inch might be erected, so as to compose a parallelepipedon on that base, of one inch high. Therefore, seeing that the content of the solid, at one inch high, is 8 cubical inches, the whole content at 5 inches high, must consequently be 5 times 8, or 40 cubical inches (since the whole solid AF may be considered, as composed of 5 such heights of cubes, one ranged above another,) And, generally, whatever the dimensions may be, it is manifest (from 21 and 22. of 7.) that the parallelepipedon will contain the cube P, as many times as the base ABCD contains the base of the cube, repeated as often as the altitude AH contains the altitude of the cube. Therefore *the content of any parallelepipedon will be found, by multiplying the area of the base by the altitude of the parallelepipedon.* Thus, for example, if the two dimensions of the base be 16 and 12 inches, and the height of the solid 10 inches; then, the area of the base being 192, the content of the solid will be 1920 cubical inches.

From the content of a parallelepipedon, thus known, *that* of a prism, or a cylinder, will likewise be known; every such solid being (by 20. 7. or 5. 8.) equal to a parallelepipedon of equal base,

and

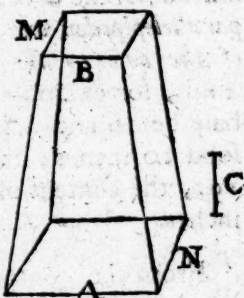
Of the Mensuration of

and altitude. Therefore, *multiply the area of the base (found by the rules for superficies) into the height of the prism, or cylinder, and the product will be the content.*

Hence the content of any pyramid, or cone, is also obtained; being (*by Cor. 3. to 8. 8.*) equal to $\frac{1}{3}$ part of a prism, or cylinder, of the same base and altitude. Therefore, *multiply the area of the base by $\frac{1}{3}$ of the altitude, and the product will be the answer.*

Every sphere being (*by 11. 8.*) equal to $\frac{2}{3}$ parts of a cylinder of the same diameter and altitude; *the content of any sphere will, therefore, be found, by multiplying the area of its greatest, or generating, circle into $\frac{2}{3}$ of its diameter: Or (because the area of such circle is to the square of the diameter, in proportion as 0,7854 to 1), let the cube of the diameter be multiplied by the fraction, 5236 ($= \frac{2}{3}$ of 0,7854), and the product will be the content.* Thus, if the measure of the diameter be 20, the cube thereof will be 8000; which, multiplied by, 5236, will give 4188,8 for the measure of the sphere's solidity.

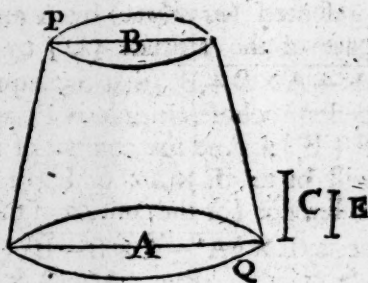
The manner of finding the content of any frustums of the solids above determined, is collected from *Theor. 10. and 11. B.VIII.* Let the frustum (MN), first proposed, be that of a pyramid; then, having found the content of a whole pyramid, of the same given base and altitude; say, as any side A of the lower end or base, is to its correspondent B of the upper, so is the said content



tent to a fourth-proportional; and, as A is (again) to B, so is the quantity last found to another proportional: which two proportionals, added to the content first determined, will give the true content of the frustum. But when the opposite bases of the frustum are squares, the rule will be more simple, and put on a better form: For then the area of the base being A^2 , the content of a whole pyramid thereon, of the same altitude with the frustum, will be equal to the parallelepipedon $C \times A^2$, C being $\frac{1}{3}$ of the given altitude of the frustum. But $A : B :: C \times A^2 : C \times A \times B$ (by 22. 7.) and $A : B :: C \times A \times B : C \times B^2$. Therefore $C \times A^2 + C \times A \times B + C \times B^2$ ($= C \times A^2 + A \times B + B^2$, by Schol. to 20. 7.) is the true content, in this case.

Hence, to find the content of the frustum of any square pyramid, add the product of the two sides of the lower and upper ends to the sum of their squares, and then multiply the aggregate by $\frac{1}{3}$ of the frustum's height.

From the content here found, that of any conical frustum (PQ) is readily obtained; being in proportion to the content ($C \times A^2 + A \times B + B^2$) of the frustum of a square pyramid cir-



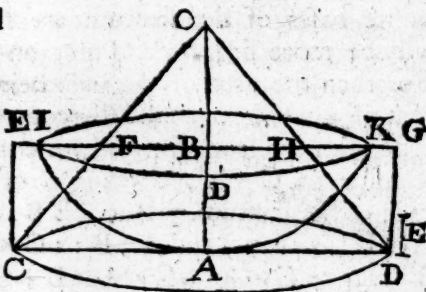
cumscribing it, as the base of the former is to the base of the latter (by Cor. 5. to 8. 8.), or as the fraction ,7854 is to unity: And so, will be equal to the ,7854. part of $C \times A^2 + A \times B + B^2 = E \times A^2 + AB + B^2$; by taking $\bar{E} = ,7854 \times C$ = the ,2618 part of the whole given altitude. Therefore,

fore, to find the content of any frustum of a cone, add the product of the diameters of the two ends to the sum of their squares; then multiply the aggregate by the frustum's height, and the product, again, by the fraction, 2618.

Hence, and from Theor. 11.

B. VIII. a Rule for finding the content of any segment IAK of a sphere, may also be deduced.

For, it appears, from thence, that



the segment proposed, IAK, is equal to the difference between a conical frustum FCDH and a cylinder ECDG of the same altitude, standing upon a base, whose radius CA is equal to that (AO) of the sphere itself. But the content of the frustum FCDH, if the two diameters CD, FH be represented (*as above*) by A and B, and the, 2618 part of the altitude (D) by E, will be $= E \times A^2 + A \times B + B^2$ (that is, equal to a parallelepipedon, whose altitude is E, and base $= A^2 + A \times B + B^2$): And the content of the cylinder ECDG will be $= 3E \times A^2$, or $E \times 3A^2$. Therefore the difference (or the content of the segment IAK) will be $= E \times 2A^2 - A \times B - B^2$ (*Schol. to 20. 7. and Ax. 5. 1.*) But $2A^2 - A \times B - B^2$ is composed of $A^2 - A \times B$ and $A^2 - B^2$; whereof the former part $A^2 - A \times B$ is $= A - B \times A$ (*by 5. 2.*) $= 2D \times A$ (because $A - B$ (or $CD - FH$) $= FE + HG = 2AB$ or $2D$); and the latter $A^2 - B^2 = A - B \times A + B$ (*by 7. 2.*) $2D \times 2A - 2D$: Whence the sum of both will consequently be $= 2D \times 3A - 2D$; and the content of the segment itself

itself $= E \times 2D \times \frac{3A - 2D}{3} = ,5236 \times D^2 \times \frac{3A - 2D}{3}$ (because $2E = ,5236D$).

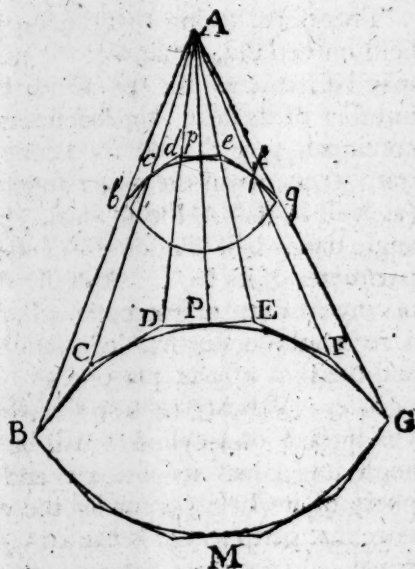
Therefore, to find the content of any segment of a sphere, multiply the square of the segment's height by the excess of thrice the sphere's diameter above the double of that height; and then multiply by the fraction ,5236.

The demonstration of the rules for determining the superficial content of the cylinder, cone and sphere, and of their several segments, or frustums, is collected from the two Lemmas here subjoined.

LEMMA 1.

The upper superficies, or area of all the sides of a regular pyramid, in which a cone may be inscribed, is equal to a rectangle under the perimeter of the base and half the length of the cone's slant side.

For, let BCDE, &c. be the base of the pyramid, and BPGM that of the inscribed cone; and from the vertex A to the point P where any side DE of the polyg. touches the circle, let AP be drawn. Then, since the triangle ADE is $= \frac{1}{2} AP \times DE = \frac{1}{2} AB \times DE$; and as the like holds good with regard to every other side



of

of the pyramid, it is evident that the sum of all the sides, or the whole superficies of the pyramid (exclusive of the base) will be equal to $\frac{1}{2}AB \times DE + EF + \mathcal{E}c$; that is, equal to a rectangle under $\frac{1}{2}AB$ and the whole perimeter of the base.

COROLLARY I.

Hence it will also appear, that all the sides of any frustum Bg of the pyramid, will be equal to a rectangle under half the length of each side and the sum of the perimeters of the two ends: For, the area of the side DEed being $= \frac{1}{2}Pp \times DE + de$, or $\frac{1}{2}Bb \times DE + de$ (by 4. 2.), the area of all the sides will, therefore, be $= \frac{1}{2}Bb \times DE + de + EF + ef + \mathcal{E}c$.

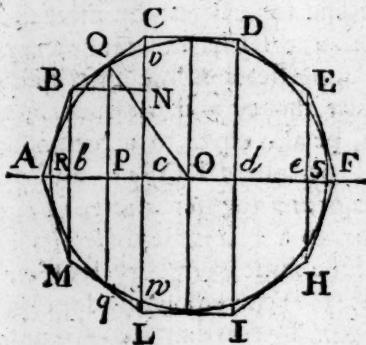
COROLLARY II.

Therefore, seeing that the foregoing conclusions hold universally, whatever the number of the sides may be; and as the pyramid, by increasing the number of its sides, approaches nearer and nearer, continually, to the inscribed cone, which is its limit; thence will the upper superficies of the cone (as well as that of the pyramid) be equal to a rectangle under half the length of its slant side and the perimeter of its base. And the convex superficies of any frustum of the cone will, *also*, be equal to a rectangle under half the length of its slant side and the sum of the peripheries of its two ends, or bases: Whence it likewise follows, that the convex surface of a cylinder will be equal to a rectangle under half its altitude and twice the periphery of its base (or under the whole altitude and once that periphery); because then the two ends are equal.—From this Corollary, the rules for finding the superficies of the cylinder and cone, are given.

LEMMA 2.

If a regular polygon ABCDE, &c. of an even number of sides, together with its inscribed circle RQSq, be supposed to revolve about the (produced) diameter RS, as an axis; the superficies of the solid generated by the polygon, will be equal to a rectangle under its axis AF and a right line equal to the circumference RQSq of the inscribed circle.

From the center O , to the point of contact Q , of any side BC , let the radius OQ be drawn; also draw BbM , QPq , CcL , &c. perpendicular to AF , and BN perpendicular to CL .



Because the solid generated by the plane $BbcC$ is the frustum of a cone, the convex superficies thereof, generated by BC is equal to a rectangle under $\frac{1}{2}BC$ and the sum of the peripheries of the two circles described by Bb and Cc , (*by Cor. 2. to the precedent*): But the sum of these two peripheries, as QP is an arithmetical mean between Bb and Cc , is equal to twice the periphery Qq ; and therefore the convex superficies of the said frustum equal to $\frac{1}{2}BC \times 2$ periph. $Qq = BC \times$ periph. Qq . But, because of the similar triangles OPQ , BNC , we have $BC : BN$ (bc) :: $OQ : PQ$:: periph. $RQSq$: periph. Qq (*by 4. 8.*); and consequently $BC \times$ periph. $Qq = bc \times$ periph. $RQSq$ = the superficies

perficies generated by BC. By the very same argument, the superficies generated by any other side CD is $= cd \times \text{periph. RQ}Sq$: Whence it is manifest, that the superficies of the whole solid is $= \frac{Ab + bc + cd + Ec}{4} \times \text{periph. RQ}Sq = AF \times \text{periph. RQ}Sq$.

COROLLARY I.

Since the superficies of the solid is, universally, equal to $AF \times \text{periph. RQ}Sq$, let the number of sides of the generating polygon be what it will; and as the said superficies, by increasing the number of sides, approaches nearer and nearer, continually, to the superficies of the inscribed sphere, which is its limit; thence will the superficies of the sphere, itself, be also equal to a rectangle under its axis RS and periphery $RQ Sq$: and the convex superficies of any segment thereof vRw , will likewise be equal to a rectangle under its axis (or height) Rc and the same periphery $RQ Sq$; since it is proved, that the corresponding superficies of CBAML, is universally equal to $Ac \times \text{periph. RQ}Sq$.

COROLLARY II.

Hence it also appears, that the superficies of every sphere is equal to four times its generating circle: Because (by 2. 8.) the circle $RQ Sq = \frac{1}{2} RS \times \frac{1}{2} \text{periph. RQ}Sq$.
 $RQ Sq = \frac{1}{4} RS \times \text{periph. RQ}Sq$.

In deriving these conclusions, as well as those depending upon the preceding Lemma, the Reader must have observed, that something is assumed, which is not demonstrated in any part of these *Elements*. But this will not, I imagine, be considered as a fault, by those who know, that it is impossible to prove in a manner perfectly regular and geometrical, that a *curve surface*, of any kind, is equal

to a *plane-one* of an assigned magnitude: Plane surfaces are compared with one another, in virtue of the 10th *Axiom*; in which, whatever relates to the equality of plane figures, has its original. But no principles have been yet admitted into the *common*, or *lower Geometry*, whereby a curve-surface can be compared with a plane-one; nor even by which the proportion of any one curve-line to a right-line can be known: Nor can it be demonstrated by all the Geometry in *Euclid's Elements*, that the periphery of a circle is less than the perimeter of its circumscribing square.—We can determine the proportion of solids bounded by curve-surfaces, by describing other solids in, and about them, so as to differ less from them, than by any assigned part however small. But in comparing of the surfaces, this method fails; because, let the number of sides of the inscribed, or circumscribed solid be ever so great, or let the solid itself approach ever so near to the proposed one; the two surfaces, after all, *will have no part in common* on which a demonstration can be formed, but will still be distinct things. Before such a comparison can possibly be made, in a regular and scientific manner, *new principles* must be laid down: But *these* belong to, and are best supplied in the *Modern Geometry*, or *Method of Fluxions*.

OF THE

MAXIMA and MINIMA

OF

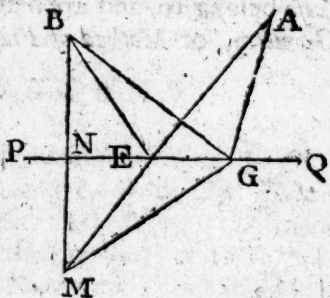
Geometrical Quantities.

THEOREM I.

If from two given points A, B, on the same side of an indefinite line PQ (in the same plane with them) two lines AE, BE be drawn to meet on, and make equal angles AEQ, BEP with the said line PQ; the lines so drawn, taken together, shall be less than any other two AG, BG, drawn from the same points to meet on the same line PQ.

For, let BNM be perpendicular to PNQ, and let AE be produced to meet it in M, also let MG be drawn.

Then the triangles MNE, BNE, having the angle MEN (= AEQ^a) = BEN^b, MNE = BNE^c, and NE com-



^a 3. 1.
^b Hyp.
^c Contr.

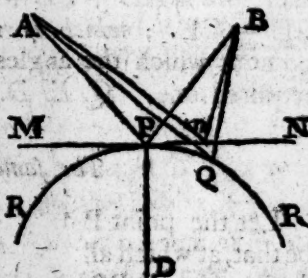
mon to both; have also $MN = BN$, and $ME = BE$

BE^a; whence also MG = BG^c: But AM (AE + ^d 15. 1.
BE) is less than AG + MG^e, or, than its equal ^e Ax. 10. 1.
AG + BG. Q. E. D. 19. 1.

THEOREM II.

Of all right-lines AP, BP; AQ, BQ, that can be drawn from two given points A, B, to meet, two by two, on the convexity of a given circle RPQR, those two AP, BP taken together, shall be the least, which make equal angles with the tangent MPN (or with the radius DB) at the point of concurrence P.

For, if to any point *n* in A the part of the tangent intercepted by AQ and BQ, there be drawn An and Bn; then will AP + BP be less than An + Bn^a, and An + Bn less than AQ + BQ^b: Consequently AP + BP is less than AQ + BQ. Q. E. D.



^aTheor. 1.
^b 23. 1.

This demonstration holds equally true, when the curve RPR is supposed of any other kind; provided all tangents to it, fall entirely without the curve.

THEOREM III.

If, in a given triangle ABC, a point is to be determined, so that the sum of all the three lines drawn from thence to the three angles, shall be the least possible; I say, the position of that point must be such, that all the angles formed about it by those lines, shall be equal among themselves.

If you deny it, then let some point E , at which the angles BEA , CEA are unequal, be the required one.

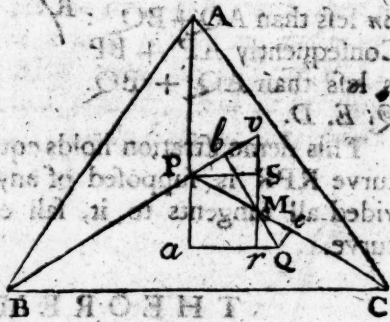
Upon the center A , thro' E , let the circumference of a circle RER be described, and let D be that point in it, where the angles ADB and ADC are equal.

¹Theor. 2. Because $BD + CD$ is less than $BE + CE$, therefore is $AD + BD + CD$ also less than $AE + BE + CE$; which is repugnant. Therefore no point at which the angles are unequal, can be the required one. *Q. E. D.*

The same otherwise.

Let the point P be that, at which all the angles APB , APC , BPC are equal*; and from any other point Q , upon the lines forming them, let fall the three perpendiculars Qa , Qb , Qc .

I say, first, that the sum of the three distances Aa , Bb , Cc , intercepted by those perpendiculars, and the three given points A , B , C , will be equal to the sum of the three



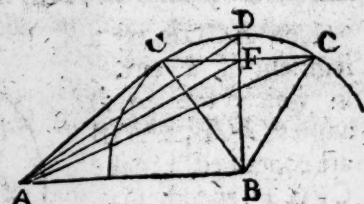
* The determination of the position of a point, at which lines drawn from three given points, shall form any given angles, is given among the Geometrical Constructions, in the next section.

first distances $AP + BP + CP$. For, if thro' the interfection M of Pc and bQ , rv be drawn parallel to Aa , meeting Bb (produced) in v , and PS , parallel to aQ , in S ; it is evident (because the angle $v = \angle BPa = \angle vPM = \frac{1}{2}$ of a right-angle^a)¹ Cor. to 8. 1. that the triangle PvM is equilateral; and that the right-angled triangles QMc , QMr , having QM ($= PMb = vMb$) $= QMr$, have also $cM = rM$; to which let $MP = Mv$ be added; so shall $cP = rv = aP + vS = aP + Pb$. And, if to the first and last of these, $AP + BP + Cc$ (be again) added; then will $AP + BP + CP = Aa + Bb + Cc$, as was asserted: Whence the Theorem itself is exceedingly obvious: For seeing that the sum $AP + BP + CP$ is but equal to the sum of the three bases Aa , Bb , Cc , it must necessarily be less than that of the three hypotenuses AQ , BQ and CQ . *Q. E. D.*

THEOREM IV.

The greatest triangle ABD that can possibly be contained under two right-lines, given in length, and any other right-line joining their extremes, will be when the two given lines AB, BD make right-angles with each other.

For, let BC be equal to BD , and the angle ABC either greater, or less than the right-angle ABD ; let also CF be drawn parallel to AB , meeting BD (produced, if necessary) in F , and let A , F , and A , C be joined.

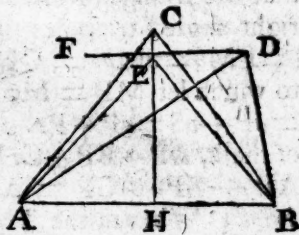


Then the angle BFC being a right-one^a, it is evident that BC (BD) is greater than BF ^o; and therefore the triangle ABD , being greater than ABF ^p, is also greater than its equal ABC . *Q. E. D.*

THEOREM V.

Of all triangles ABC , ABD , having the same base AB , and the sum of their other sides the same, the isosceles one ACB , is the greatest.

Let CH be perpendicular to AB , and DEF parallel to AB , intersecting HC (produced if need be) in E ; likewise let AE and BE be drawn.

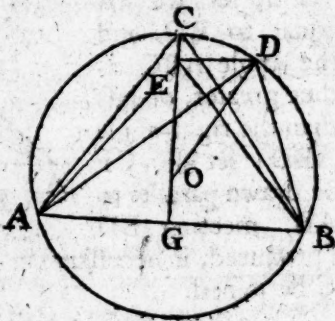


It is manifest that the angles AEF , BED are equal¹⁶; therefore $AE + BE$ is less than $AD + BD$ ⁷, or than its equal $AC + BC$ ¹; and so the triangle AEB , falling within the triangle ACB ¹, must be less than ACB ¹; and therefore $ADB (= AEB)$ must also be less than ACB . $\therefore E. D.$
¹⁶ 16. and
⁷ 7. 1.
¹ Theor. 1.
¹ Hyp.
¹ 23. 1.
¹ Ax. 2. 1.
¹ Cor. 1.
 to 2. 3.

THEOREM VI.

Of all triangles ABC , ABD standing upon the same base AB , and having equal vertical angles ACB , ADB , the isosceles one ACB is the greatest.

Let $ACDB$ be a segment of a circle, in which the equal angles ACB , ADB are contained²²; make CEG perpendicular, and DE parallel, to AB ; from the center O draw OD , and let A, E and B, E be joined. It is evident



that CG , not only bisects AB ¹⁶, but also passes through the center O ¹. Therefore, OD (OC) being

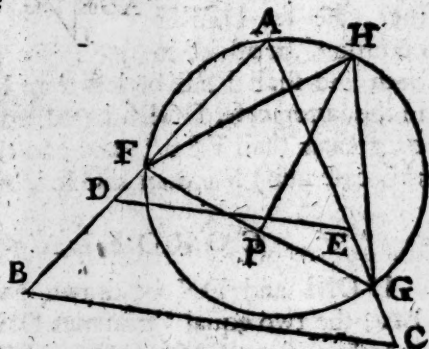
¹⁶ 16. 1.
¹ Cor. to
 2. 3.

ing greater than OE^a , the triangle ACB will also ^a 20. 1.
be greater than AEB , or than its equal ADB^b . ^b Cor. 2. 2.
Q. E. D. and Ax. 2.

THEOREM VII.

Of all right lines DE , FG that can be drawn to cut off equal areas ADE , AFG from a given triangle ABC , that DE is the least, which makes the triangle ADE , cut off, an isosceles one.

Let AFG be the circumference of a circle passing through the three points A , F , G ; also let PH be perpendicular to FG , at the middle point P , meeting the circumference in H , and let FH

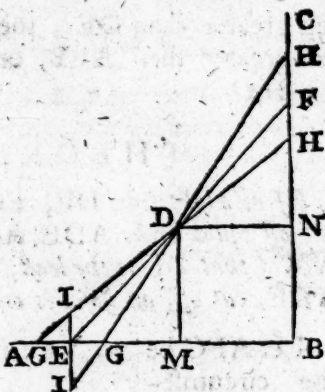


and GH be drawn. The triangle FHG , being isosceles^c, is therefore greater than FAG^d , or ^e Ax. 10. than its equal^e ADE : Whence, as the triangles ^d Theor. 6. FHG , ADE are equiangular^f, the base FG of the ^f Hyp. greater, must consequently exceed the base DE of ^g Hyp. and Cor. 10. the lesser. Q. E. D. 10. 1.

THEOREM VIII.

Of all right-lines EF , GH , GH that can be drawn thro' a given point D , between two right-lines BA , BC given in position; that EF which is bisected by the given point D , forms with them the least triangle (EBF).

- For, if EI, parallel to BC, be drawn, meeting GH in I; the^b equiangular triangles DFH and DEI, having DF = DEⁱ, will be equal^k; and DFH will therefore be less, or greater than DEGⁱ, according as BG is greater or less than BE. In the former case let DEBH
- ^a 3. and 7. 1. ⁱ Hyp. ^k 15. 1. ¹ Ax. 2. ² Ax. 6.
- common, be added to both; so shall FEB be less than HGB^k. And if, in the latter case, DGBF be added, then will HGB be greater than FEB^m; and consequently FEB (in this case also) less than HGB. Q. E. D.



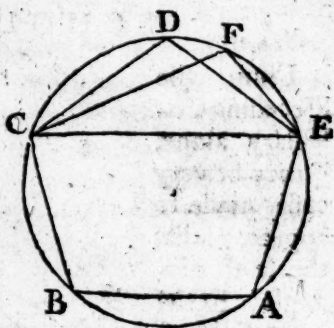
COROLLARY.

- If DM and DN be drawn parallel to BC and BA; the two equal^e triangles DEM, DFN, taken together (since $EM = DN^e = MB^p$) will be equal to the parallelogram DMBN^q; and consequently the parallelogram DMBN = $\frac{1}{2}$ BEF $\sqsupset$ $\frac{1}{2}$ BGH. Whence it is manifest, that a parallelogram is always less than half a triangle in which it is inscribed, except when the base of the one is half the base of the other; in which case the parallelogram is just half the triangle.
- ^e 15. 1. ^p 24. 1. ^q Cor. to 2. 2.

SCHO-

For, if possible, let some polygon $ABCFE$, whose sides CF , EF are unequal, be the greatest.

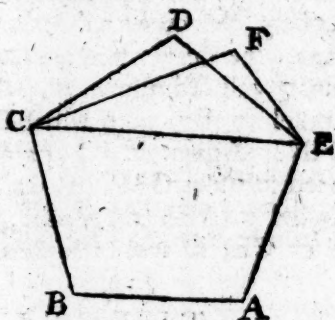
Let CDE be an isosceles triangle described in the same segment with CFE ; which being greater than CFE , the whole polygon $ABCDE$ will also be greater than the whole polygon $ABCFE$; which is repugnant. Therefore the polygon is the greatest when the sides are all equal.



THEOREM X.

Of all right-lined figures, contained under the same perimeter, and number of sides, the greatest is, when the sides are all equal.

For, if $ABCDE$ be the greatest possible, the triangle CDE must, manifestly, be greater than any other triangle CFE upon the same base, whereof the sum of the other sides is also the same. But, by Theo-



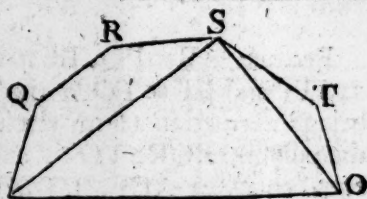
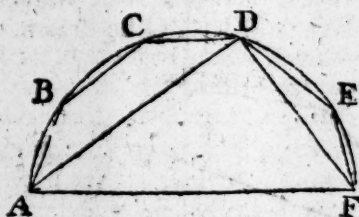
rem V. the greatest triangle, when the base and the sum of the sides are given, is that whose sides are equal: Therefore DC and ED are equal. In the same manner it appears that $BC = CD$, &c. $\therefore E D$.

THEO-

THEOREM XI.

If all the sides of a polygon, except one, be given in length, and their position be required, so as to make the polygon itself the greatest possible; I say, their position must be such, that two lines drawn from the extremes of the unknown side to any angle of the polygon, shall form a right-angle.

For, if you would have the polygon ABCDEF to be the greatest possible, and yet ADF, subtended by the unknown side AF, not a right-angle: Then let PSO be a right-angle, contained under $PS = AD$, and $OS = FD$; and upon PS and OS, let the figures PSRQ and OST be described equilateral, and equal, to ADCB and FDE.



described equilateral, and equal, to ADCB and FDE.

to. 6.

The triangle PSO is greater than ADF; therefore, PSRQ being = ADCB, and OST = FDE, the whole polygon PQRSTO is also greater than the whole polygon ABCDEF, which is repugnant. Ax. 6. 1.

Theor. 4.
Confr.

COROLLARY.

Hence, because the angle in a semi-circle is a right-angle; it appears, that the greatest polygon that can be contained under any proposed number of given sides, and one other side any how taken, will be, when it may be inscribed in a semi-circle, whereof the indetermined line will be the diameter.

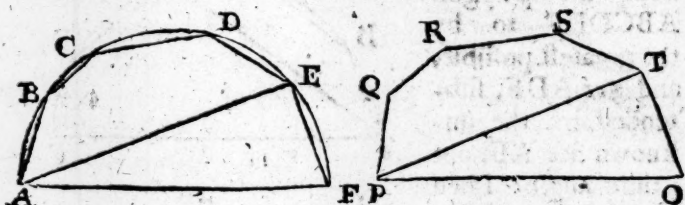
13. 3.

THEQ.

THEOREM XII.

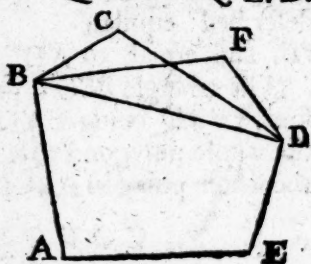
A polygon ABCDEA in a circle, is greater than any other polygon PQRSTP, whatever, whose sides are the same both in length and number.

Let AF be the diameter of the circle, and join E, F; also make the angle $\angle PTO = \angle AEF$, $TO = EF$, and let PO be drawn.



- Because $AB = PQ$, $BC = QR$, $CD = RS$, $DE = ST$, and $EF = TO$, the polygon ABCDEF, being inscribed in a semi-circle, will be greater than the polygon PQRSTO; and, if from these, the equal triangles AEF, PTO be taken away, there will remain $ABCDEA > PQRSTP$. $\therefore E. D.$
- Hyp.
 • Theor.
 11.

That the magnitude of the greatest polygon, which can be contained under any number of unequal sides, does not at all depend upon the order in which those lines are connected to each other, will appear,



thus. Let ABCDE be the greatest, one way, or according to one order of the sides; and upon BD let a triangle BDF be constituted whose sides DF and BF are, respectively, equal to BC and DC; then, the triangles BCD, BFD being equal, the whole

whole polygons ABCDE and ABFDE will likewise be equal, notwithstanding their equal sides BC, DF, &c. are placed according to different orders.

THEOREM XIII.

Of all polygons, contained under the same perimeter, and number of sides; that whose sides, and angles, are equal, is the greatest.

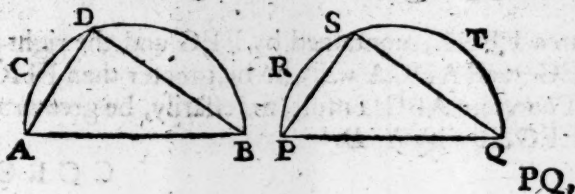
For, the greatest polygon that can be contained under a given perimeter, is one whose sides are all equal^a. But, of all the polygons of this sort, that is the greatest which may be inscribed in a circle^b; Therefore, the greatest of all, is that whose sides are all equal, and which may be inscribed in a circle, or whereof the angles, as well as the sides, are all equal. Q. E. D.

^a Theor.
^{10.}
^b Theor.
^{12.}

THEOREM XIV.

The greatest area than can possibly be contained by one right-line, any how taken, and any other line or lines, whatever, whereof the sum is given; will be, when two right-lines drawn from the extremes of the unknown line first mentioned, to meet any where in the given boundary, make right-angles with each other.

For, if you would have the area ACDEBA, contained by some right-line AB, and ACDEB whereof the length is given, to be the greatest possible, and ADB, at the same time, not a right-angle: Then, let PSQ be a right-angle, contained under $PS = AD$, and $QS = BD$; and, having joined



- PQ, upon PS and QS conceive two figures PRS and QST to be formed, equal, and alike in all respects to ACD and DBE. Since the area PSQ is greater than ADB^c; it is manifest, that the area PRSTQP, contained by the right-line (PQ) and PRSTQ (= ACDEB^d) will also be greater than the area ACDEBA^e; *which is repugnant*: Therefore the area ACDEBA cannot be the greatest possible, unless the angle ADB be a right one.
- Q. E. D.

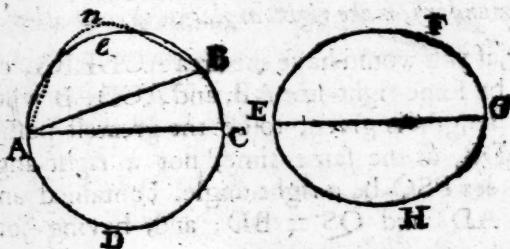
COROLLARY.

- Hence, because the angle in a semi-circle is a right-angle^f, it is evident that the area will be the greatest possible, when the given length, or boundary, forms the arch of a semi-circle; whereof the indetermined right-line proposed is the diameter.

THEOREM XV.

Of all plane figures ABCD, EFGH, contained under equal perimeters (or limits), the circle (ABCD) is the greatest.

For, if the diameter AC be drawn, and EFG be taken equal to the arch ABC; then the area ABCE will (by the precedent) be greater than the



area EFGE, contained by EFG and the right-line EG; and ADCE will also be greater than FHGE: Therefore ABCD must, necessarily, be greater than EFGH. Q. E. D.

COROL-

COROLLARY.

Hence it appears, that the greatest area that can possibly be contained by a right-line AB, and a curve-line A α B, both given in length; will be, when the latter is an arch of a circle. For, let A α B be any other curve-line, equal to A α B, and let the whole circle A α BCD be completed; which will (it is proved) be greater than the mixed figure A α BCD; and consequently, by taking away the common segment ABCD, there will remain A α BA greater than A α BA.

THEOREM XVI.

The greatest parallelepipedon that can be contained under the three parts of a given line AB, any how taken, will be when all the parts are equal to each other.

For, if possible, let two parts AE, ED A $\xrightarrow{\quad C \quad E \quad D \quad}$ B be unequal. Bisect AD in C; then will the rectangle under AE (AC+CE) and ED (AC-CE) be less than AC² (or AC $\times$ CD) by the square of CE². Therefore the solid AE $\times$ ED $\times$ DB will also ^{7. 2.} be less than the solid AC $\times$ CD $\times$ DB³; which is ^{21. 7.} contrary to hypothesis.

COROLLARY.

Hence, of all rectangular parallelepipedons, having the sum of their three dimensions the same, the cube is the greatest.

THEO.

THEOREM XVII.

The greatest parallelepipedon $AC^2 \times CB$ that can possibly be contained under the square of one part AC of a given line AB and the other part CB , any how taken, will be, when the former part is the double of the latter.

For, let Ac and Bc be any other $A \xrightarrow{d, P} \xrightarrow{c, C} B$ parts, into which

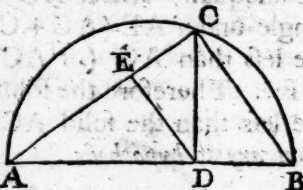
- ¹ Cor. to the given line AB may be divided; and let AC and Ac be bisected in D and d . So shall $AC^2 \times CB = 4AD \times DC \times CB^1 \square 4Ad \times dc \times cB^2 (AC^2 \times cB^1)$
^{6. 2. and} by the precedent. $\mathcal{Q}. E. D.$

THEOREM XVIII.

The hypotenuse AB of a right-angled triangle ABC being given; the solid $BC \times AC^2$ contained under one leg BC and the square of the other AC , will be the greatest possible, when the square of the latter leg AC is double to that of the former BC .

For, if CD be conceived perpendicular to AB , and DE to AC ; it will be $AC^2 (AB \times AD^1)$:

- ¹ Cor. to $AB^2 :: AD : AB^m :: DE : BC^n$; and consequently, $AC^2 \times BC = AB^2 \times DE^o$;
^{19. 4.}
^{m 21. 7.}
^{n 14. 4.}
^{o 23. 7.}



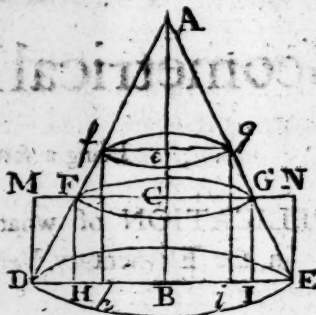
- which (as AB^2 is given) will, evidently, be the greatest possible, when DE , or its square $^p (DE^2)$ is the greatest possible. But $DE^2 : AD^2 :: ^q BC^2 (BD \times AB^1) : AB^2 :: BD : AB^m$; and therefore $DE^2 \times AB = AD^2 \times BD^o$; which (and consequently DE^2) will be the greatest possible, when AD is the double of BD^1 ; that is, when $AC^2 (AD \times AB)$ is the double of $BC^2 (BD \times AB)$. $\mathcal{Q}. E. D.$

THEO-

THEOREM XIX.

The altitude BC of the greatest cylinder HG that can possibly be inscribed in any cone ADE, is one third part of the altitude AB of the cone, and the cylinder itself $\frac{2}{3}$ parts of the cone.

For, let gh be any other cylinder inscribed in the cone; and it will be, $AC^2 \times BC : CG^2 \times BC :: AC^2 :: CG^2 :: AC^2 : cg^2 :: AC^2 \times Bc : cg^2 \times BC$; whence, by alternation, $AC^2 \times BC : AC^2 \times Bc :: CG^2 \times BC : cg^2 \times Bc$: and so likewise is the cylinder HG



22. 7.
Cor. 11.
4.

to the cylinder gh ; but $AC^2 \times BC$ is greater than $Ac^2 \times Bc$; therefore HG is also greater than gh . Again, since $AC = \frac{2}{3}AB$, and therefore $CG = \frac{2}{3}BE$; we also have, cylinder HG : cone ADE (or cylinder DN) :: $CG^2 (\frac{2}{3}BE^2) : BE^2 :: \frac{4}{9} : 1$. and 3. 8. Q. E. D.

3 and
5. of 8.
Theor.
17.
Ax. 4.
Cor. 3.
to 8. 8.
21. 7.
and 3. 8.

SCHOLIUM.

From this proposition, by reasoning as in the Scholium to Theorem VIII. it will appear, that the least cone that can be described about, and the greatest cylinder that can possibly be described in, any solid generated by the rotation of a curve, concave to its axis, will be, when the sub-tangent is two-thirds of the altitude of the cone, or twice the altitude of the cylinder; and that the two figures will be in the ratio of nine to four. From whence the dimensions of the greatest and least cylinders and cones, that can be described in, and about solids generated by curves, to which the method of drawing tangents is known, may be readily determined.

P

THE

THE CONSTRUCTION

Of a great Variety of

Geometrical Problems.

Being a farther

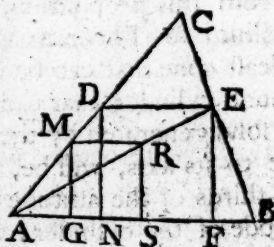
APPLICATION of what has been delivered
in the Elementary Part of this Work.

PROBLEM I.

In a given triangle ABC, to inscribe a square
DEFN.

CONSTRUCTION.

FROM any point M,
in either side, upon
the base AB, let
fall the perpendicular MG;
make MR perpendicular,
and equal, thereto, and let
ARE be drawn, meeting
the other side of the tri-
angle in E; then draw ED
parallel, and EF and DN perpendicular, to AB;
and the thing is done.



DE-

DEMONSTRATION.

Let RS be drawn parallel to EF: Then (*by similar triangles*) $RS : (MG) : EF :: AR : AE :: MR : DE$: Therefore, as MG and MR are equal, *by construction*, EF and DE will likewise be equal.

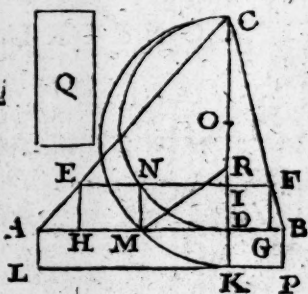
By the same method a rectangle may be inscribed in a triangle, whose sides shall be in a given ratio; if MR and MG (instead of being equal) be taken in the given ratio; the rest of the construction being exactly the same.

PROBLEM II.

In a given triangle ABC, to inscribe a rectangle EFGH equal to any given right-lined figure Q, not exceeding half the triangle.

CONSTRUCTION.

On the base AB (*by* 7. 6.) let a rectangle ABPL be constituted = Q; and let LP meet the perpendicular CD of the triangle (produced) in K. Then (*by* 17. 5.) let CD be divided in I, so that $CI \times DI = CD \times DK$ (that is, let two semi-circles be described on CD and CK; drawing MN and NI parallel to CD and AB): So shall DI be the altitude of the required rectangle.



DEMONSTRATION.

Since (*by Constr.*) $CI \times DI (= NI^2 = MD^2) = CD \times DK$, thence will $DI : DK :: CD : CI :: AB : EF$ (*by* 20. 5.); and consequently $DI \times EF$ (*by* 10. 4.) = $AB \times DK = Q$. Q. E. D.

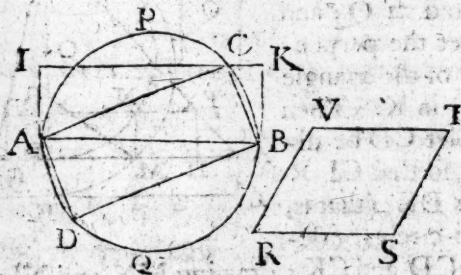
The Construction of

That the Problem will be impossible, when Q is greater than half the triangle, is evident from the Construction, as well as from the Theorem on p. 200. It may also be observed, that there is another way, besides that used above, for dividing CD in the manner proposed; which (though not more obvious) is, in point of conciseness, rather preferable; and is thus. Having (*as before*) found a mean-proportional DM between CD and DK , and bisected CD in O ; from M to CD apply $MR = OD$, and take $OI = RD$. So shall $CI \times DI = OD^2 = OI^2$ (*by 7.2.*) $= MR^2 = RD^2$ (*by Hyp.*) $= DM^2 = CD \times DK$ (*as before*).

PROBLEM III.

In a given circle APBQ, to inscribe a rectangle equal to a given right-lined figure RSTU, not exceeding half the square of the diameter.

CONSTRUCTION.



Upon the diameter AB describe the rectangle $ABKI = RSTU$ (*by 7.6.*); and from the point C , where the side KI intersects the periphery of the circle, draw CA and CB , parallel to which draw BD and AD ; then will $ACBD$ be the rectangle that was to be constructed.

The Construction of

PROBLEM V.

Between two lines AB, AC, given by position, to apply a line KL, equal to a given line MN, so that the triangle AKL formed from thence, shall be of a given magnitude.

CONSTRUCTION.

Having bisected MN in D, on MD describe a rectangle MDEF (by 7. 6.) = the magnitude given: also on MN let a segment of a circle be described (by 22. 5.) to contain an angle $\hat{=}$ A; and from its intersection with EF, draw HM and HN; then make AK = HM, AL = HN, and join K, L. So shall the base KL be also = the base MN, and the triangle AKL equal, and like in all respects, to HMN; which last (by Cor. to 2. 2.) is, manifestly, equal to the magnitude given MDEF. Q. E. D.

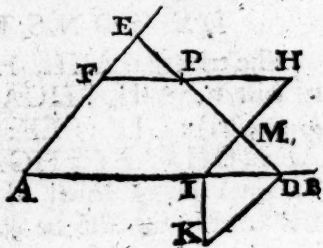


PROBLEM VI.

Through a given point P, to draw a line EPD to meet two lines AB, AC, given by position, so that the triangle ADE formed from thence, shall be of a given magnitude.

CONSTRUCTION.

Draw FPH parallel to AB, intersecting AC in F; and on AF, let a parallelogram AFHI be formed, equal to the given area of the triangle: Make IK perpendicular to AI, and equal



to FP; and from K, to AB, apply $KD = PH$; then draw DPE, and the thing is done.

DEMONSTRATION.

The triangles PHM, PFE and MDI, by reason of the parallel lines, are similar; and therefore, since the three homologous sides PH (KD), FP (IK) and DI are such, as to form a right-angled triangle (*by Constr.*), the triangle PHM on the first of them, is equal to both the other two PFE and MDI (*by 29. 4.*): and, if to these equal quantities, AFPM be added; then will AFHI be also ADE.

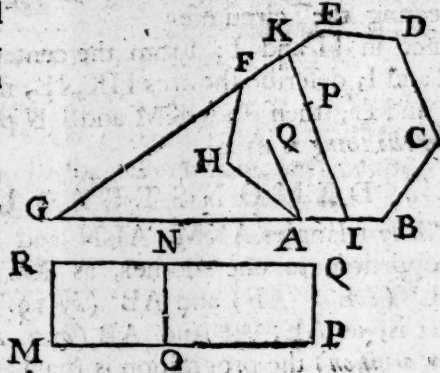
This Problem will be impossible, when KD (PH) is less than KI (PF); that is, when the area given is less than a parallelogram under AF and 2FP.

PROBLEM VII.

From a given polygon ABCDEFH, to cut off a part AIKFH, equal to a given rectangle MN; by line (IK), either parallel to a given line AQ, or passing through a given point P.

CONSTRUCTION.

Let BA and EF be produced to meet in G; and upon ON let a rectangle OQ be constituted (*by 7. 6.*) equal to AGFH; then, by Prob. the 4th, or 6th, according to the case proposed,



The Construction of

draw IK, so as to make the triangle $GKI = MQ$, and the thing is done.

The Demonstration whereof is manifest from the Construction.

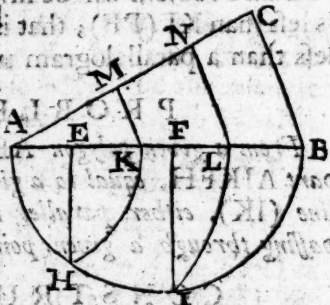
And, in the same manner, the polygon may be divided according to any given ratio; because, the whole being given, each part will be given.

PROBLEM VIII.

To divide a given triangle ABC into any proposed number of parts (AKM, KN, LC) so as to have any given proportion to each other, by means of lines drawn parallel to one of the sides BC of the triangle.

CONSTRUCTION.

Let AB be divided into parts, AE, EF, FB, having the same given proportion to each other, as the parts of the triangle are to have. Upon AB let a semi-circle AHIB be described, and perpendicular to AB, draw EH, FI, meeting the circumference in H and I: From the center A, through H and I, describe the arcs HK, IL, meeting AB in K and L; then draw KM and LN parallel to BC, and the thing is done.



DEMONSTRATION.

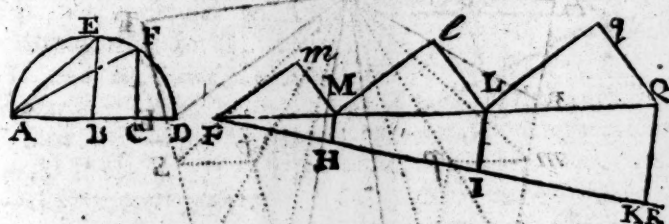
The triangles AKM, ALN and ABC, are in proportion, to one another, as AK^2 ($AB \times AE$), AL^2 ($AB \times AF$) and AB^2 (by 19. and 24. 4.); that is, as AE, AF and AB (by 7. 4.) Whence (by division) the proposition is manifest.

PROBLEM IX.

To divide a given line PQ into any proposed number of parts, so that similar right-lined figures PMm, ML, LQq, described upon them, shall have the same given ratio among themselves, as an equal number of right-lines AB, AC, AD assigned.

CONSTRUCTION.

Upon the greatest AD of the given lines AD, AC, AB, describe a semi-circle AEFD; and perpendicular to AD, draw BE and CF, meeting the circumference in E and F; and, having drawn



PR, at pleasure, in it take PH = dist. AE, HI = dist. AF, and IK = AD; draw KQ, and parallel thereto draw HM, IL; which will divide PQ, as required.

DEMONSTRATION.

$PMm : LQq :: PM^2 : LQ^2$ (by 26. 4.) $:: PH^2$ ($= AE^2 = AB \times AD$, by 19. 4.) $: IK^2$ ($= AD^2$) $:: AB : AD$ (by 7. 6.). In the very same manner it appears, that $ML : LQq :: AC : AD$. Q. E. D.

PRO-

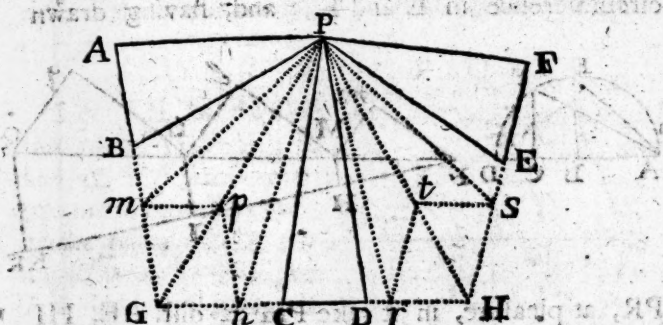
The Construction of

PROBLEM X.

To determine the position of a point P , so that lines drawn from thence to the extremes of three right-lines AB , CD , EF , given in length and position, shall form three triangles ABP , CPD , EPF , mutually equal to each other.

CONSTRUCTION.

Let the given lines be produced to meet in G and H ; in which take $Gm = AB$, $Hr = EF$, and



Gn , Hr , equal each to CD : Complete the parallelograms $Gmpn$, $Hrts$; and let the diagonals Gp and Ht be produced till they meet in P , and the thing is done.

DEMONSTRATION.

Let PA , PB , Pm , &c. be drawn. The triangles GPn , GPm , having the same base GP , and equal altitudes (because $Gmpn$ is a parallelogram) are therefore equal to each other: But $CPD = GPn$, and $APB = GPm$ (by Cor. to 2. 2.); whence CPD and APB are likewise equal. And, from the very same reasoning, it appears that CPD and FPE are equal. Q. E. D.

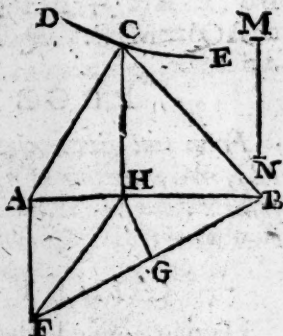
P R O.

PROBLEM XI.

From two given points A, B, to draw two lines AC, BC, to meet in a line DE of any kind, given by position; so that the difference of their squares shall be equal to a square given (MN^2).

CONSTRUCTION.

Make AF perpendicular to the line AB, and equal to MN; draw BF, which bisect with the perpendicular GH; and from its intersection with AB, draw HC perpendicular to AB, meeting DE in C; draw AC and BC, and the thing is done.



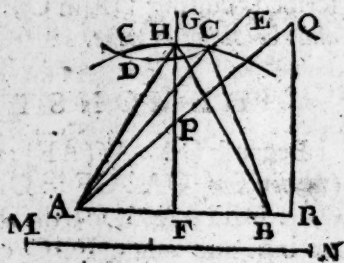
For $BC^2 - AC^2 = BH^2 - AH^2$ (by 9. 2.) $= FH^2 - AH^2 = AF^2 = MN^2$.

PROBLEM XII.

From two given points A, B, to draw two lines AC, BC, to meet in a line DE of any kind, given by position; so that the sum of their squares shall be equal to a given square, MN^2 .

CONSTRUCTION.

Bisect AB with the perpendicular FG, in which take $FP = FA$, and draw $APQ = MN$; on AB (produced, if necessary) let fall the perpendicular QR; from A to FG apply $AH = AR$, and



about

The Construction of

about the center F, thro' H, let the circumference of a circle be described; and from its intersection C with the given line DE, draw CA and CB, and the thing is done.

DEMONSTRATION.

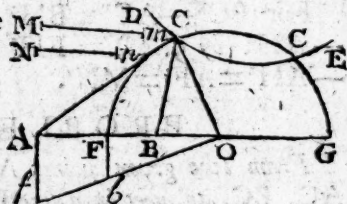
Let BH be drawn; which being $= AH = AR = RQ$ (because $FP = AF$); thence will $AC^2 + BC^2 = AH^2 + BH^2$ (by 20. 3.) $= AR^2 + RQ^2 = AQ^2 = MN^2$. Q. E. D.

PROBLEM XIII.

From two given points A, B, to draw two lines AC, BC, meeting in a line DE of any kind, given by position; so as to obtain the ratio of two unequal right-lines Mm, Nn assigned.

CONSTRUCTION.

Having joined the M given points, divide AB in F (by 15. 5.) so that $AF : BF :: Mm : Nn$; make Af and Fb parallel to each other, taking the former $= AF$, and the latter $= FB$; and thro' their extremes draw fbO, meeting AB, produced, in O; from whence, with the radius OF, let the semi circle FCG be described, cutting DE in C; then draw AC and BC, and the thing is done.



DEMONSTRATION.

Because $OA : Af(AF) :: OF : Fb(FB)$, we have (by division) $OA : OF :: OF : OB$; or $OA : OC :: OC : OB$. And so, the triangles OAC, OCB, having one angle FOC common, and the sides about it proportional, they must, therefore, be similar

similar (*by* 15. 4.); whence the other sides will also be proportional, or $AC : BC :: OA : OC (OF) :: Af : Fb :: Mm : Nn$ (*by Constr.*) 2. E. D.

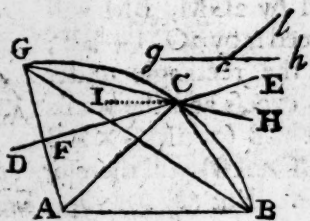
Note. When, in either of the two preceding Problems, the circle described, neither cuts nor touches the given line DE, the thing proposed to be done, will be impossible; as no two lines drawn from A and B, to meet above the circumference, can possibly have their ratio, or the sum of their squares, the same as two lines meeting in the circumference.

PROBLEM XIV.

From two given points A, B, to draw two lines AC, BC, meeting in a right-line DE, given by position; so as to make therewith two angles ACD, BCE, whose difference shall be equal to an angle given, bcb .

CONSTRUCTION.

Make AFG perpendicular to DE, and $FG = AF$; and, having drawn GB, on it let a segment of a circle GCB be described (*by* 22. 5.) to contain an angle equal to the supplement (beg) of the given one bcb ; and from its intersection (C) with DE, draw CA and CB; and the thing is done.



For, if BC and GCH be drawn, then will $ACD = GCD = ECH = BCE - BCH$ (bcb).

In the same manner, the Problem will be constructed, when, instead of the difference of ACD and BCE, that of ABC and BAC is given: Because, when DE is parallel to AB, the latter difference

The Construction of

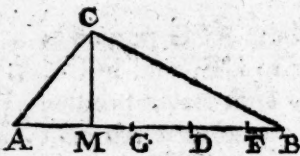
ference is equal to the former; and, in all other cases, differs from it by twice the given angle GCI, expressing the inclination of the said lines.—When the sum of the angles ACD and BCE is given, the angle ACB is also given: And here, nothing more is necessary, than barely to describe, upon AB, a segment of a circle to contain the said given angle ACB.

LEMMA.

If, of any three proportional lines AB, DB, FB, the difference AF of the two extremes be bisected in G; and if on the greatest AB, as a base, a triangle ABC be so formed, that its lesser side AC shall be to the distance MG of the perpendicular from the bisecting point G, in the given ratio of AB to DB; then shall the greater side BC exceed the lesser AC by the given line DB.

DEMONSTRATION.

Because FM exceeds AM by $2GM$, BM will exceed it by $2GM + BF$; and the rectangle under this excess and the whole base AB ($= 2MG \times AB + BF \times AB$) will therefore (by 9. 2.) be $= BC^2 - AC^2$. But (by hypothesis and 10. 4.) $MG \times AB = AC \times BD$, and $BF \times AB = BD^2$: Therefore $2AC \times BD + BD^2 = BC^2 - AC^2$; and, by adding AC^2 , common, $AC^2 + 2AC \times BD + BD^2$ (or the square of $AC + BD$, by 6. 2.) will be $= BC^2$; and, consequently, $AC + BD = BC$. Q. E. D.



This Lemma is not only of use in the Problem next following, but will be found a ready instrument in the solution of many others; for which reason it is here put down.

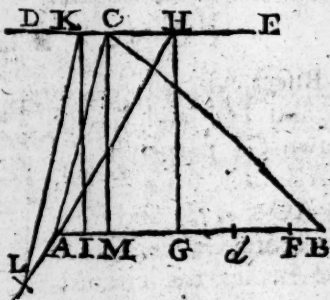
PRO-

PROBLEM XV.

From two given points A, B, to draw two lines AC, BC to meet in a right-line DE, given by position; so that their difference shall be equal to a given right-line Bd.

CONSTRUCTION.

In AB, take a third-proportional BF to BA and Bd; and, having bisected AF in G, take $GI = Bd$; make GH and IK perpendicular to AB, meeting DE in H and K; and draw HAL, to which, from K, apply KL =



AB; and parallel thereto draw AC, meeting DE in C; join B, C; *and the thing is done.*

DEMONSTRATION.

Let CM be perpendicular to AB.

Then, because of the parallel lines, it will be $AC : KL (AB) :: HC : HK :: GM : GI (Bd)$. Whence (*by alternation*) AC being to GM , as BA to Bd ; and AB , dB , FB being also proportionals; the whole Construction is manifest from the Lemma premised.

If the sum, instead of the difference, of the two lines (AC, BC) be given, the method of construction will be exactly the same, without the least alteration of any one step; provided that Bd' be first of all taken (in BA, produced) equal to the given sum, instead of the difference.

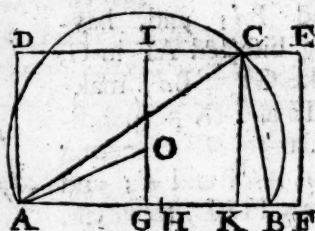
PRO-

PROBLEM XVI.

From two given points, A, B , to draw two lines AC, BC , so as to meet in a right-line DE parallel to that (AB) joining the said points, and that the rectangle $(AC \times BC)$ contained by them, shall be equal to a rectangle given, $ADEF$.

CONSTRUCTION.

Bisect AB and AF , in G and H ; and, having drawn GI perpendicular to AB , to it, from A , apply $AO = AH$; and from the center O , through A and B , let the circumference of a circle be de-



scribed intersecting DE in C ; from whence draw CA and CB , and the thing is done.

For, if CK be drawn perpendicular to AB , it is evident (by 25. 3.) that $AC \times BC = CK \times 2AO = AD \times AF$. Q. E. D.

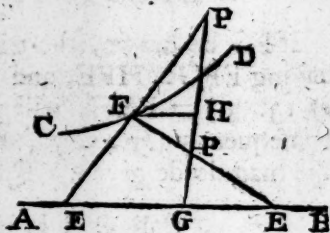
PROBLEM XVII.

Through a given point P , so to draw a line FPE that the parts thereof PF, PE , intercepted by that point and two lines AB, CD , given in position, shall obtain a given ratio.

CON.

CONSTRUCTION.

Through P, to any point in AB, draw PG, in which (by 13. 5.) take PH to PG, in the given ratio of PF to PE; draw HF parallel to AB, meeting CD in F; then draw FPE, and the thing is done.



For, the triangles PGE, PHF being equiangular (by 3. and 7. 2.) thence is $PF : PE :: PH : PG$: and so PF and PE (as well as PH and PG) are in the ratio given.

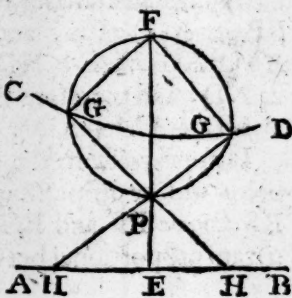
In this construction, it is necessary that one of the two given lines should be a *right one*: The other may, it is manifest, be a line of *any kind* whatever.

PROBLEM XVIII.

Through a given point P, to draw a line GH terminating in a right-line AB, and in a line CD of any kind, given both by position; so that the rectangle under the two parts thereof PG, PH, shall be of a given magnitude.

CONSTRUCTION.

Having drawn EPF perpendicular to AB, take therein PF (by 7. 6.) so that $PE \times PF =$ the magnitude given: Upon PF, as a diameter, let a circle be described, intersecting CD in G: then draw GPH, and the thing is done.



Q

DE.

DEMONSTRATION.

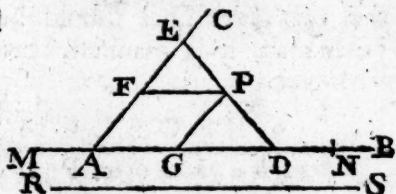
If FG be drawn, the triangles PFG and PHE, having $FPG = HPE$, and $FGP (= \text{a right-angle, by 13. 3.}) = PEH$, will be equiangular: And, consequently (by 24. 3.) $PG \times PH = PF \times PE =$ the magnitude given. Q. E. D.

PROBLEM XIX.

Through a given point P, between two right-lines AB, AC, given by position, so to draw a line ED, that the sum of the segments (AD + AE) cut off by it from the two former, shall be equal to a given line RS.

CONSTRUCTION.

Draw PF and PG parallel to AB and AC; in BA produced take $AM = AF$, and $MN = RS$; then divide GN in D (by 17.5.) so that $GD \times ND = AM \times AG$; draw DPE, and the thing is done.



DEMONSTRATION.

By similar triangles, $GD : GP (= AF = AM) :: FP (= AG) : FE$; and consequently $GD \times FE = AM \times AG = GD \times ND$ (by Constr.): Whence $FE = ND$; and therefore $FE + AF + AD (AE + AD) = ND + AM + AD = MN = RS$. Q. E. D.

It appears from the Construction, that the Problem will be impossible, when (GN) the excess of RS above PF and PG, is less than the double of a mean-proportional between these two quantities.

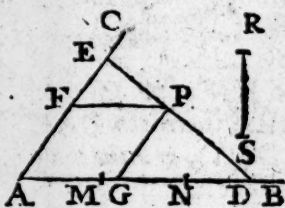
P R O-

PROBLEM XX.

Through a given point P, between two right-lines AB, AC, given by position, so to draw a line DE, that the difference $(AD - AE)$ of the segments cut off by it from the two former, shall be equal to a given line RS.

CONSTRUCTION.

Draw PF and PG parallel to AB and AC; in AB take $AM = AF$, and $MN = RS$; then (by 18. 5.) let a line ND be added to GN, so that $GD \times ND = PG \times PF$; draw DPE, and the thing is done.



DEMONSTRATION.

Because of the similar triangles PGD, EFP, we have $GD \times FE = PG \times PF = GD \times ND$ (by Constr.) and consequently $FE = ND$; whence $AD - AE = AN - AF (AM) = MN = RS$. Q. E. D.

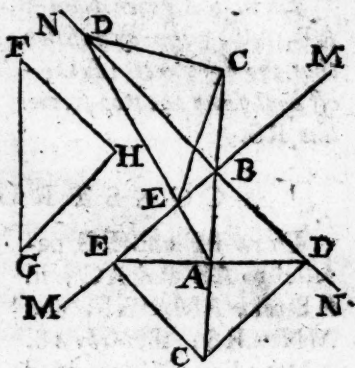
PROBLEM XXI.

Between two right-lines NBN, MBM, given by position, to apply a line DE of a given length, and which (produced, if necessary) shall pass through a given point A, equally distant from the said given lines.

The Construction of

CONSTRUCTION.

Let FG be the length given; and, having drawn AB , make the angles F, G equal each to ABD (or ABE); and in AB produced take AC (by 18. 5.) so that $AC \times BC = HG^2$: from C to NN apply $CD = HG$; then draw DAE (DEA), and the thing is done.



DEMONSTRATION.

Because $AC : CD :: CD : BC$ (by Constr. and 10. 4.) the triangles CAD and CDB (having one angle common, and the sides about it proportional) will be equiangular (by 15. 4.): And therefore, since $CDA = DBC = ABE$ (by hypothesis), the circumference of a circle may be described through all the four points, C, D, B, E (by 11. 3. or 19. 3.). And so the angle DEC , standing on the same subtense (DC) with DBC , will be equal to it; and, consequently, equal to EDC . Therefore the isosceles triangles EDC, FGH being equiangular, and having $CD = HG$, their bases ED and FG will also be equal. Q. E. D.

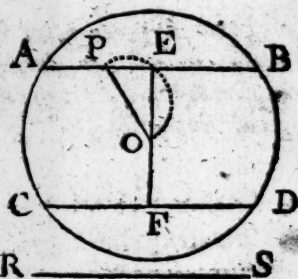
PROBLEM XXII.

In a given circle $ABDC$, to apply a chord AB equal to a given line RS (less than the diameter) and which shall pass through a given point P .

CON-

CONSTRUCTION.

Inscribe $CD = RS$, upon which, from the center O , let fall the perpendicular OF ; also draw OP , upon which let a semi-circle be described, and in it apply $OE = OF$; lastly, through P and E , let AB be drawn; which will, manifestly, be equal to $CD (= RS$, by 3. 3.) as being equally distant from the center, by construction.



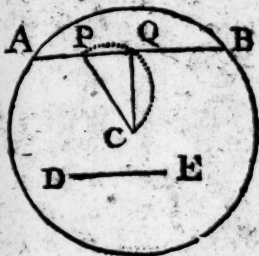
When the point given is placed without the circle, the construction will be no-ways different.

PROBLEM XXIII.

Through a given point P, so, to draw a line AB, that the parts thereof AP, BP, intercepted by that point and the circumference of a given circle, shall have a given difference, DE.

CONSTRUCTION.

From the center C , draw CP , upon which let a semi-circle PQC be described, and in it apply PQ , equal to half DE , producing the same, both ways, to meet the circumference in A and B : So shall $AQ = BQ$ (by 2. 3.); and therefore $PB (= BQ + PQ) = AQ + PQ = AP + 2PQ = AP + DE$, which was to be done.



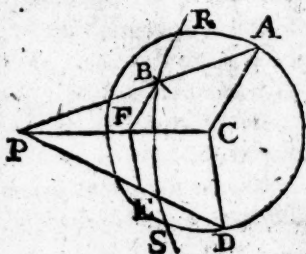
The Construction of

PROBLEM XXIV.

From a given point P, to the circumference of a given circle C, to draw a right-line PBA, so as to be divided in a given ratio by a line RBS of any kind, given in position.

CONSTRUCTION.

To any point D in the circumference, draw PD, which divide at E in the ratio given; and, having drawn PC and the radius CD, parallel to the latter draw EF, meeting the former in F; from whence, to RS, apply $FB = FE$; then through B draw PA, and the thing is done.



DEMONSTRATION.

Let CA be drawn. Then, because of the parallel lines CD, EF, it will be, $CD (AC) : EF (FB) :: PC : PF$; and so the triangles PAC, PBF will likewise be equiangular (by 16. 4.). And therefore $PB : BA :: PF : FC :: PE : ED$ (by 13. 4.). Q. E. D.

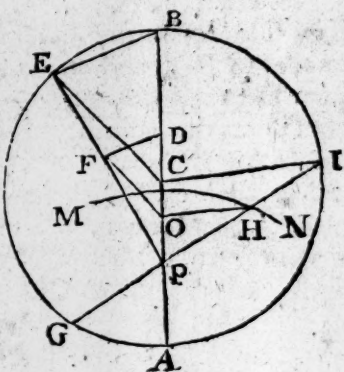
PROBLEM XXV.

Through a given point P, to draw a line GH, terminating in the circumference of a given circle BGA, and in a line MN of any kind, given by position; so that the rectangle under the two parts thereof PG, PH, shall be of a given magnitude.

CON-

CONSTRUCTION.

Through P draw the diameter AB, in which let PD be so taken (*by 7. 6.*) that $PA \times PD =$ the magnitude given. From any point E in the circumference, draw EP and the radius EC; and, having joined BE, draw DF parallel to BE, and from its intersection with PE, draw FO parallel to EC, meeting PB in O; from whence, to MN, apply $OH = OF$; and through P draw HG for the line required.



DEMONSTRATION.

Let PH be produced (if necessary) to meet the circumference of the circle in I; and let C, I be joined.

The lines OF, CE being parallel, thence will $PO : PC :: OF (OH) : CE (CI)$; and therefore OH and CI will likewise be parallels (*by 16. 4.*) Therefore $PH : PI :: PO : PC :: PF : PE :: PD : PB$; whence, alternately, $PH : PD :: PI : PB :: PA : PG$ (*by 21. 3. and 10. 4.*) and consequently $PH \times PG = PD \times PA$. Q. E. D.

PROBLEM XXVI.

From the circumference of a given circle C, to a line MN of any kind, given by position, so to draw a right-line EF, as to be both equal and parallel to a given right-line PQ.

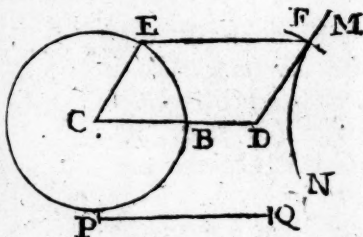
Q 4

CON-

The Construction of

CONSTRUCTION.

From the center C, draw CD equal and parallel to PQ; and, from D to MN, apply DF = the radius CB; draw CE equal and parallel to DF; then, EF being drawn, it will (by 26. 1.) be equal, and parallel to CD; which was to be done.

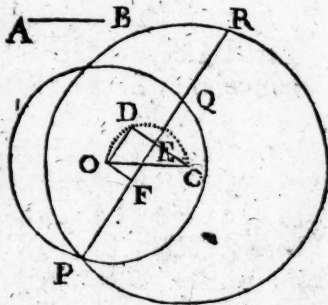


PROBLEM XXVII.

From the point of intersection P of two given circles O and C, so to draw a line PR, that the part thereof QR intercepted by the two peripheries, shall be equal to a given line AB.

CONSTRUCTION.

Upon the line OC joining the two centers, let a semi-circle ODC be described, in which apply $OD = \frac{1}{2}AB$; and parallel thereto, draw PR, and the thing is done.



DEMONSTRATION.

Let CD, and OF parallel to CD, be drawn, meeting PR in E and F. Then, the angle ODC being a right one (by 13.3.) and PR parallel to DO, E and F will also be right-angles, and $EF = DO$ (by 24.1.): And so, $PE - PF$ being $(= DO) = \frac{1}{2}AB$, it is manifest, that $2PE (PR) - 2PF (PQ) = AB$, or that $QR = AB$. Q. E. D.

§

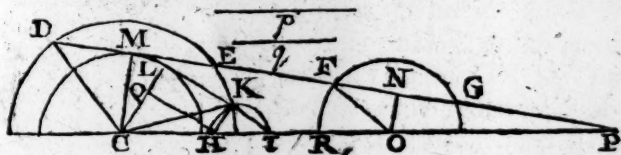
PRO-

PROBLEM XXVIII.

From a given point P, in the line passing through the centers of two given circles C and O; so to draw a line PD, that the parts thereof DE, FG, intercepted by those circles, shall be in a given ratio, (viz. as p is to q.)

CONSTRUCTION.

Take CH to OR, as p to q; and CI to OR, as PC to PO: Upon HI let a semi-circle be described, intersecting the circle CDE in K; through



which point draw IKL, and make CL perpendicular thereto; at which distance, from the center C, let a circle ML be described: Then, if from P a line PD be drawn to touch this circle, the thing is done.

DEMONSTRATION.

Let CD, CK, HK, FO be drawn, and also CM and ON, perpendicular to PD, and HQ to CL.

The right-angled triangles CDM, CKL having $CD=CK$, and $CM=CL$, have also $DM=LK=QH$ (since, by 13. 3. the angle HKI, as well as L, is a right-one). Moreover (by Construction) $CI:OR (OF)::PC:PO::CM (CL):ON$; and so the triangles CIL, OFN (having their sides proportional) must be similar; and consequently OFN also similar to CHQ; whence, as QH (or DM):FN::CH:FO (OR)::p:q (by Constr.).
Q. E. D.

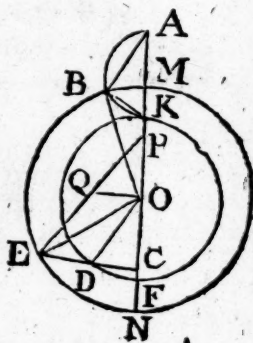
PRO.

PROBLEM XXIX.

To draw a line EC, to make given angles with a line MN passing through the common center O, of two given circles MEN, KDF, so that the parts thereof CD, ED, intercepted by that line and the two peripheries, shall obtain a given ratio.

CONSTRUCTION.

Let QOM be the given angle, to which ECM shall be equal: In OM produced, let KA be so taken, as to be to the radius OK in the given ratio of ED to CD; upon which let a segment of a circle ABK be described to contain an angle equal to QOM; and, from its intersection with the circle MEN, draw BA; parallel to which, draw the radius OD; and then, through D, draw EC parallel to QO.



DEMONSTRATION.

Let BK, BO, EO be drawn, and also EP parallel to DO, meeting OQ and OM in Q and P.

Then, the triangles AKB, POQ, having $ABK = POQ$ (by Construction), and $KAB = OPQ$ (by 7. 1.) have their external angles OKB, EQO also equal (by 9. 1.): Therefore, because $EQ (=OD) = OK$, and $EO = BO$, thence is $OQ = KB$ (by 17. 1.); and therefore, as the triangles POQ, AKB are equiangular, PQ is also $= AK$: But (because of the parallel lines) $CD : ED :: DO (OK) : PQ (AK)$. Q. E. D.

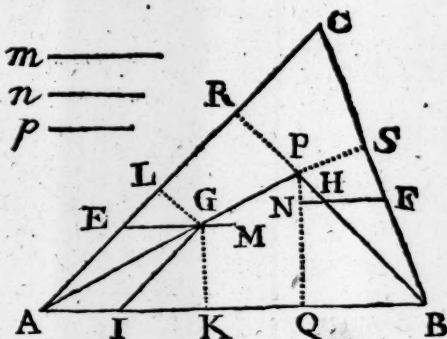
PRQ.

PROBLEM XXX.

To determine a point P, so that three perpendiculars drawn from thence, to as many right-lines AB, AC, BC, given by position, shall obtain the ratio of three given lines m, n, and p, respectively.

CONSTRUCTION.

Take AE and BF each $= m$, and draw EM and FN parallel to AB; in which take $EG = n$, and $FH = p$; thro' G and H draw AP and BP, and the point of concurrence P, will be that which was to be determined.



DEMONSTRATION.

Upon the sides of the triangle ABC let fall the perpendiculars GL, GK, PR, PQ, PS; and let GI, parallel to AC, be drawn.

The angle LEG being $= LAK = GIK$ (Cor. 1. to 7. 1.) and L and K being both right-angles (by Constr.), the triangles EGL and GIK are similar; and therefore $IG (m) : EG (n) :: GK : GL :: PQ : PR$ (by 21. 4.). And, in the very same manner, $m : p :: PQ : PS$. Q. E. D.

After the same way, the Problem may be constructed, when the lines drawn from P are required to make any given angles with the lines upon which they fall.

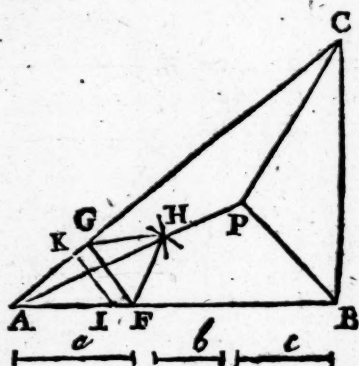
P R O.

PROBLEM XXXI.

To determine a point P, so that three lines PA, PB, PC, drawn from thence to three given points A, B, C, shall obtain the ratio of three given lines a, b, and c, respectively.

CONSTRUCTION.

Having joined the given points, in AB take $AF = a$, and $AI = c$; make the angles AFG and AIK equal, each, to $\angle ACB$; and from the centers F and G, with the radii b and AK, let two arcs be described, intersecting in H; from which point draw HF



and HA; then draw BP to make the angle $ABP = AHF$, and it will meet AH (produced) in the point P, required,

DEMONSTRATION.

Let BP, CP, and GH be drawn.

The triangles ABP, AHF being equiangular, it will be, $AP : BP :: AF (a) : FH (b)$; also $AB : AP :: AH : AF$; and $AB : AC :: AG : AF$ (because ABC and AGF are likewise equiangular, (by Constr.). Now, seeing the extremes of the two last proportions are the same, the four means, AP, AC, AG, AH (by 10. 4.) will therefore be proportionals; and so, the triangles ACP, AHG being equiangular (by 15. 4.) it will be $AP : CP :: AG : GH (AK) :: FA (a) : AI (c)$. Q. E. D.

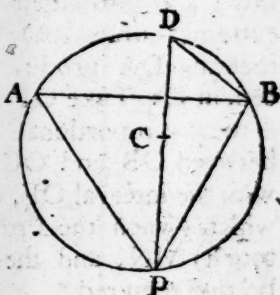
PRO.

PROBLEM XXXII.

To determine the position of a point P , at which, lines drawn from three given points A, B, C , shall make given angles, one with another.

CONSTRUCTION.

Draw AB , upon which (by 22.5.) let a segment of a circle APB be described, capable of containing the given angle which the lines drawn from A and B are to include; and let the whole circle be completed; make the angle ABD equal to that which the lines drawn from A and C are to include; and from the point D , where BD meets the circumference, through C , let DP be drawn, meeting the circumference in P ; which is the point required.



For, AP and BP being drawn, the angle APC will be $=$ the given angle ABD (by 11.3.) both standing on the same arch AD : And APB is also of the given magnitude by construction.

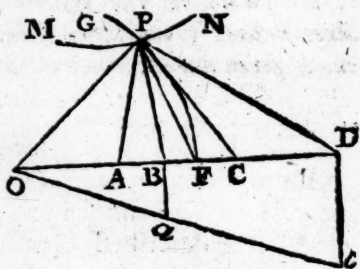
PROBLEM XXXIII.

Any two unequal segments AB, CD of a right-line AD being given, as well in position as length; to determine a point P in a line MN of any kind, given by position; at which the two angles APB, CPD , subtended by those segments, shall (if possible) be equal to each other.

CON-

CONSTRUCTION.

Make Ba , and Dc parallel to each other, taking the former = BA , and the latter = DC ; and thro' their extremes draw caO , meeting DA produced, in O : Take OF a mean-proportional



between OB and OC ; and from the center O , with the interval OF , let a circle FG be described; which (when the Problem is possible) will cut (or touch) MN , and the point of intersection P , will be *that* required.

DEMONSTRATION.

Let PO , PA , PB , PF , PC , and PD be drawn.

Because $OD : Dc (DC) :: OB : Ba (BA)$, it will be (*by division*) $OD : OC :: OB : OA$; and consequently $OD \times OA = OC \times OB = OP^2$ (*by Constr.*) Therefore, seeing that $OA : OP :: OP : OD$, and that the angle O is common to both the triangles OAP , OPD , these triangles must be equiangular (*by 15. 4.*) and consequently $APF = OPF$ (OFP) — OPA (ODP) = DPF (*by 9. 1.*). In the very same manner, because $OB : OP :: OP : OC$, the angle BPF will be = CPF ; and, consequently, APB also = CPD . *Q. E. D.*

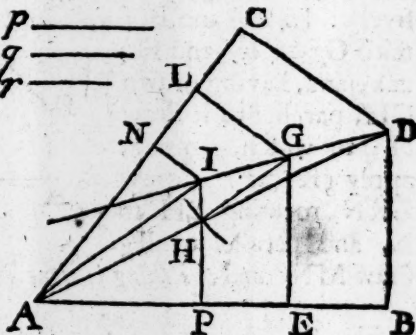
PRO.

PROBLEM XXXIV.

Two right-lines AB, AC being given, both in length and position; from the point of their concurrence A, so to draw another line AI, that two perpendiculars, IP, IN, falling from the extreme thereof, upon the two given lines, shall cut off alternate segments BP, CN in a given ratio each to the line AI so drawn.

CONSTRUCTION.

Let the given ratio of AI, BP and CN be that of the lines p , q and r , respectively. Take $BE = q$, and $CL = r$; making BD , EG , CD and LG perpendicular to AB and AC , so as



to meet in D and G: draw DA and DG, and from G to AD, apply $GH=p$, and parallel thereto draw AI, meeting DG produced (if needful) in I, *and the thing is done.*

DEMONSTRATION.

Because of the parallel lines, $AI : GH :: ID : GD :: BP : BE$; whence, alternately, $AI : BP :: GH(p) : BE(q)$. In the very same manner, $AI : CN :: p : r$. *Q. E. D.*

PRO-

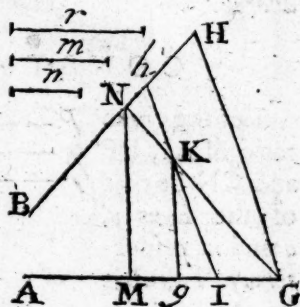
The Construction of

PROBLEM XXXV.

Between two lines AG, BH, given both in position and length, to draw a line MN, which shall be in a given ratio to each of the segments MG, NH, cut off from the two given lines.

CONSTRUCTION.

Let the given ratio of MN, MG and NH be that of r, m and n , respectively: In AG and BH take $Gg = m$, and $Hh = n$; and, having drawn GH, parallel to it draw bl ; to which, from g , apply $gK = r$; draw GKN, meeting BH in N, and parallel to Kg draw MN, and the thing is done.



DEMONSTRATION.

Because of the parallel lines, it will therefore be $MN : GM :: gK (r) : Gg (m)$; and likewise $MN : gK (r) :: GN : GK :: NH : Hh (n)$; which last (by alternation) is $MN : NH :: r : n$.
Q. E. D.

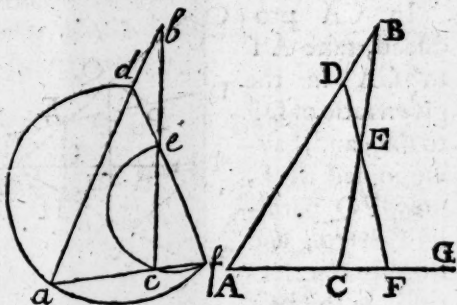
PRO.

PROBLEM XXXVI.

To draw a line DF , to cut three other lines AB , AG , BC , given by position; so that the two parts thereof DE , EF , intercepted by those lines, shall be respectively equal to two given lines, de , ef .

CONSTRUCTION.

Upon the right-lines df , ef let two segments of circles, daf , ecf , be described, to contain angles, respectively equal to BAG and BCG : Then



(by Prob. 27.) draw fa , so that the part ac , intercepted by the two peripheries, shall be equal to AC ; join a , d , and take $AF = af$, $AD = ad$; then draw DF , and the thing is done.

DEMONSTRATION.

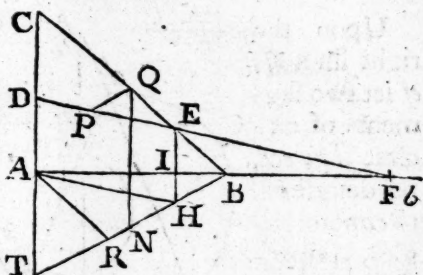
The triangles FAD , fad , having $AF = af$, $AD = ad$, and $A = a$ (by Construction), are equal in all respects; and therefore (if ce be drawn) the triangles FCE , fce , having $F = f$, $FCE = fce$, and $CF = cf$, will also be equal and alike: Therefore, seeing the wholes DF , df , and the parts FE , fe are equal, the remaining parts DE and de must likewise be equal. Q. E. D.

PROBLEM XXXVII.

Through a given point P, to draw a line DPF, to cut three lines AC, CB, ABb, given by position; so that the parts thereof DE, EF intercepted by those lines, shall obtain a given ratio.

CONSTRUCTION.

In CA produced, take AT to CA in the given ratio of DE to EF; and, having joined B, T, draw PQ parallel thereto; and from its intersection with BC, draw QN parallel to CT; also draw AR parallel to BC: And in RB take RH (by 18. 5.) so that $NH \times RH = PQ \times TR$: Then draw AH, and DPEF parallel to it, and the thing is done.



DEMONSTRATION.

Because (by Constr. and 10. 4.) $NH : TR :: PQ : RH :: QE : AR$ (by 14. 4.) it follows (by alternation) that $NH : QE :: TR : AR :: TB : BC$ (by 12. 4.) $:: BN : BQ$: Therefore EH (when drawn) will be parallel to QN or AD (by Cor. to 12. 4.); and so DEHA being a parallelogram, we have again (by similar triangles) as DE (AH); $EF :: IH : IE :: AT : AC$. Q. E. D.

When the segments AD, BE, cut off from the given lines AC, BC, are required to be in a given ratio (instead of DE and EF), the construction will be

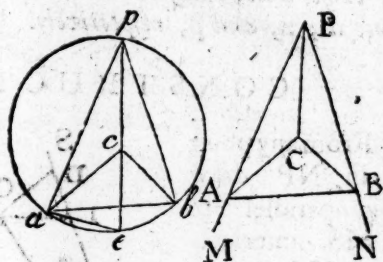
The Construction of

PROBLEM XXXIX.

To draw two lines CA, CB, from a given point C, to terminate in two other lines PM, PN, given by position, and which, together with the line AB, joining their extremes, shall form a triangle, ABC, similar to a given one, abc.

CONSTRUCTION.

Upon *ab* let a segment of a circle *apb* be described, to contain an angle equal to *MPN*, and let the whole circle be completed; draw *PC*, and also *ae*, making the angle *bae* = *CPN*, and intersecting the periphery in *e*; and through *c* draw *ep*, meeting the periphery in *p*; make the angles *PCA* and *PCB* respectively equal to *pca* and *pcb*; then join *A, B*, and the thing is done.



DEMONSTRATION.

If *pa* and *pb* be drawn; then will the angle *bae* (*CPB*) = *cpb*, both standing on the same arch *be*; therefore, *APB* being also = *apb* (*by Constr.*), the remaining angles *APC* and *apc* must consequently be equal; whence, as *PCA* = *pca*, and *PCB* = *pcb* (*by Constr.*), the triangles *APC*, *apc*, and *BPC*, *bpc* are equiangular; and therefore *AC* : *ac* ($\because PC : pc$) :: *CB* : *cb*. And so the triangles *ABC*, *abc*, having the sides about the equal angles *ACB*, *acb*, proportional, they are like to each other (*by 15. 4.*). Q. E. D.

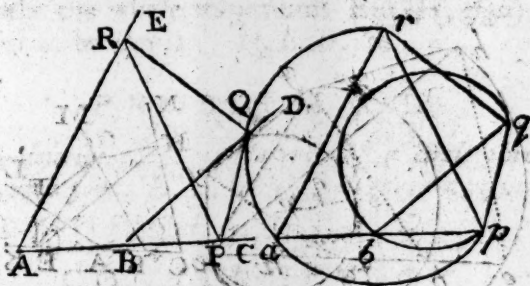
P R O.

PROBLEM XL.

To describe a triangle PQR , equal and similar to a given triangle pqr , which shall have its angular points placed in three right-lines ABC , BD , AE , given by position.

CONSTRUCTION.

Upon the two sides pq , pr , let two segments of circles pbq , par be described, to contain angles respectively equal to CBD and CAE . Then draw pa (by Prob. 27.) so that the part thereof ba , intercepted by the two peripheries, shall be equal to



BA ; and, having joined b , q , and a , r , make $BP = bp$, $BQ = bq$, $AR = ar$, and let PQ , PR , and QR be drawn for the sides of the triangle required.

DEMONSTRATION.

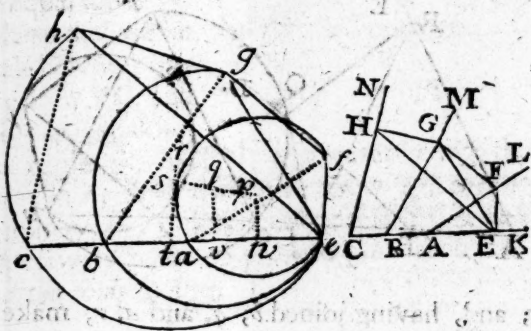
The triangles PAR , par ; PBQ , pbq , being equal and alike in all respects (by Constr.), not only the sides PR , pr ; PQ , pq , but the angles QPR , qpr , will be equal; and, consequently, the two triangles PQR , pqr also equal and like to each other. Q. E. D.

PROBLEM XLI.

To describe a trapezium similar to a given one $efgh$, having its angular points placed in four right-lines CN , BM , AL , $CBAK$ given by position.

CONSTRUCTION.

Let three points p , q , r be found, from whence, as centers, segments of circles may be described, upon ef , eg , eh , to contain angles equal to the three given angles KAL , KBM , KCN , respectively; draw pp , in which, produced (if necessary) take qs to qp in the proportion of BC to AB ;



and, having drawn rst , make ec perpendicular thereto, intersecting the three circles in a , b , c . Take $AE : AB :: ae : ab$; and make the angles CEH , CEG , and CEF , respectively equal to ceb , ceg , and cef ; then let H , G and G , F be joined; and, I say, the trapezium $EFGH$ will be similar to the given one $efgh$.

DEMONSTRATION.

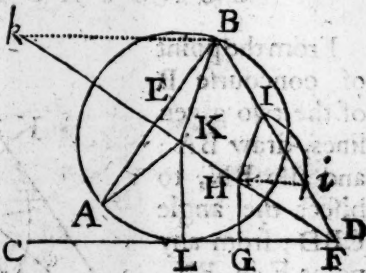
Let af , bg , ch , be drawn, and also pw , qv , perpendicular to ec . So shall $ab = eb = ea = 2ev = 2ew = 2pw$, and $bc = ec = eb = 2et = 2ev = 2vt$. And therefore $ab : bc :: vw : vt :: pq : qs$ (by 13. 4.) $:: AB : BC$ (by Constr.). Again, seeing the triangles EAF , eaf , EBG , ebg , ECH , ech , are respectively, equiangular (by Constr.) it will be $EG : eg :: EB : eb :: AE : ae$ (by Constr. and Composition) $:: EF : ef$ (by 14. 4.), and so the triangles GEF , gef (having the sides about the equal angles proportional) are similar. And, in the very same manner, the two remaining triangles EGH , egh (and consequently the whole trapeziums $EFGH$, $efgb$) will appear to be similar. Q. E. D.

PROBLEM XLII.

To describe the circumference of a circle through two given points A , B , which shall touch a right-line CD given by position.

CONSTRUCTION.

Draw AB , which bisect in E by the perpendicular EF , meeting CD in F ; from any point H in EF , draw HG perpendicular to CD ; and, having drawn BF , to the same apply $HI = HG$, and parallel thereto draw BK , meeting EF in K ; then from the center K , with the radius BK , let a circle be described, and the thing is done.



DEMONSTRATION.

Join K, A, and draw KL perpendicular to CD : Then, because of the parallel lines, $HG : HI :: KL : KB$ (by 21. 4.) ; whence, as HG and HI are equal, KL and KB are equal likewise : But it is evident, from the Construction, that KA is $= KB$; therefore $KB = KL = KA$. Q. E. D.

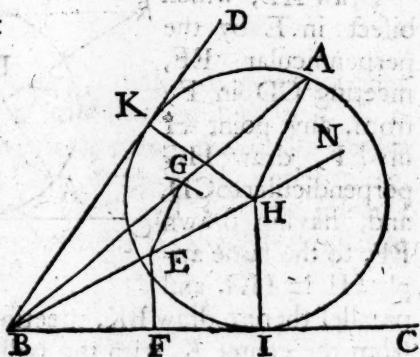
Because two equal lines HI, H_i may be applied from H to BF (except, when $\angle H$ is a right-angle) the Problem will therefore admit of two solutions. But when a perpendicular let fall from H upon BF is greater than HG ; the Problem will be impossible. And the like is to be understood in the construction of the subsequent Problems.

PROBLEM XLIII.

To describe the circumference of a circle through a given point A, which shall touch two right-lines BC, BD given by position.

CONSTRUCTION.

From the point of concurrence B of the two given lines, draw BA ; and also BN, to bisect the angle CBD ; from any point E in BN, upon BC, let fall the perpendicular EF, and to BA apply $EG =$



EF,

EF, parallel to which draw AH, meeting BN in H; then from the center H, with the interval HA, let a circle be described, and the thing is done.

DEMONSTRATION.

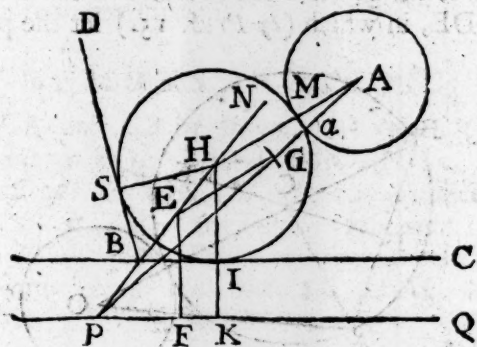
Upon BC and BD let fall the perpendiculars HI and HK; which are manifestly equal, because (by Constr.) the angle $HBI = HBK$: Moreover, as EF and EG are equal, HI and HA are also equal (by 21. 4.). $\angle E. D.$

PROBLEM XLIV.

To describe a circle, which shall touch a given circle AaM, and two right-lines BC, BD, given by position.

CONSTRUCTION.

Draw PQ parallel to BC, at the distance of the radius Aa; and through the point of concurrence B of the two given lines, draw NBP, bisecting the angle CBD, and meeting PQ in P; moreover, from any point E in PN, upon PQ, let fall the



perpen-

The Construction of

perpendicular EF, and from the same point, to PA, apply $EG = EF$; draw AH parallel to EG, meeting the periphery of the given circle in M, and the right-line PN in H, from which last point, as a center, through M, describe the circumference of a circle; and the thing is done.

DEMONSTRATION.

Draw HS perpendicular to BD, and HK to PQ, intersecting BC in I.

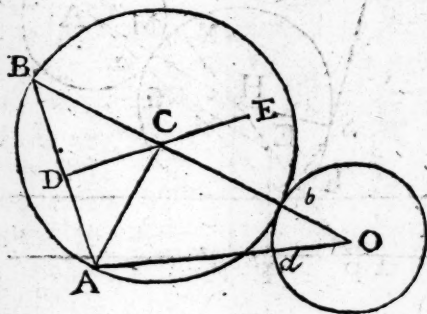
Then, because EG and EF are equal (*by Constr.*), HA and HK (*by 21. 4.*) are likewise equal; from which take away $KI = AM$ (*Aa*), and the remainders HI, HM will be equal: But, it is evident, that HI is $= HS$, because BH bisects the angle IBS; therefore $HI = HM = HS$.
Q. E. D.

PROBLEM XLV.

To describe the circumference of a circle through two given points A, B, and which shall also touch another circle Odb, given in position and magnitude.

CONSTRUCTION.

Bisect the given distance AB with the perpendicular DE, in which (*by Prob. 15.*) let the point C



be

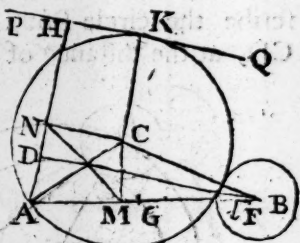
be so determined, that CO (when drawn) shall exceed CA by the radius *Od* of the the given circle. Then that point, it is manifest, will be the center of the circle to be described.

PROBLEM XLVI.

Through a given point A, to describe the circumference of a circle, which shall touch a given circle B, and also a right line PQ given by position.

CONSTRUCTION.

Make AH perpendicular to PQ, and BD to AH; and having drawn AB, in it take BF a third-proportional to AB and the radius *Bd*, and let AF be bisected in G: then draw MN (*by Prob.* 35.) so that MN, NH, and MG may be in the same given ratio, among themselves, as BD, BA, and *Bd*: and at M and N let two perpendiculars be erected on AB and AG; which will intersect each other in the center C of the required circle.



DEMONSTRATION.

Let CA and CB be drawn, and also CK perpendicular to PQ. Because $MN : NH :: BD : BA$ (*by Constr.*) $:: MN : AC$ (*by 22. 4.*), thence is $NH (CK) = AC$. And since (*by Constr.*) $NH (AC) : MG :: BA : Bd$, it is also manifest, from the Lemma on p. 222. that $BC = AC + Bd$. Q. E. D.

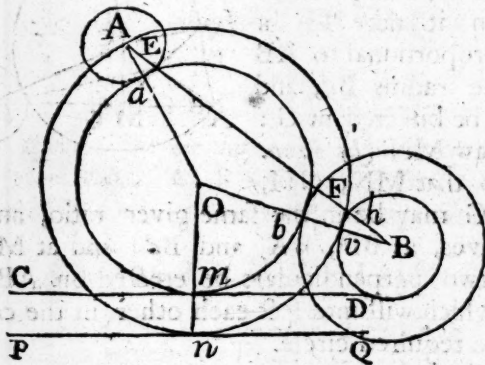
PRO-

PROBLEM XLVII.

To describe a circle $Oamb$, which shall touch two given circles AEa , BFb , and also a right-line CD , given by position.

CONSTRUCTION.

From the radius BF of the greater circle, take away Fb equal to the radius AE of the lesser; and from the center B , with the interval Bb , describe the circle Bbv ; also draw PQ parallel to CD , at the distance of Fb or AE : Then, by the



last Problem, let the center O of a circle be found, whose circumference shall pass thro' A , and touch PQ and Bbv ; and the same point O will, likewise, be the center of the required circle $Oamb$.

DEMONSTRATION.

Draw On , perpendicular to PQ , cutting CD in m ; also let OA and OB be drawn intersecting the circles

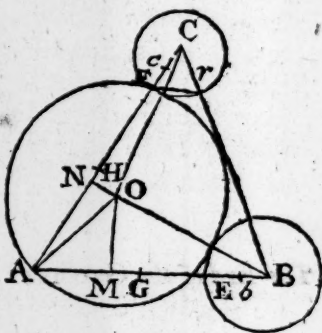
circles A and B in q and v : Then, since (*by Constr.*) $AQ = vO = nO$, and $Aa (AE) = bv (Fb) = nm$, let these last be respectively taken from the former, and there will remain $Oa = Ob = Om$. *Q. E. D.*

PROBLEM XLVIII.

To describe the circumference of a circle through a given point A, which shall touch two other circles B and C, given in position and magnitude.

CONSTRUCTION.

To the centers of the given circles, draw AB and AC; in which take $Bb =$ a third-proportional to AB and the radius BE; and $Cc =$ a third-proportional to AC and the radius CF: bisect Ab and Ac in G and H, and let Fr be drawn parallel to AB, meeting BC in r . Then (*by Prob. 34.*) draw AO, so that OM and ON being drawn perpendicular to AB and AC, the three lines AO, GM and HN shall have the same given ratio among themselves, as AB, BE and Fr. Then shall the point O be the center of the required circle: For, since (*by Constr.*) $AO : GM :: AB : BE$; and $AO : HN (:: AB : Fr) :: AC : CF$; it is manifest, from the Lemma on p. 222. that $OB = AO + BE$; and $OC = AO + CF$. *Q. E. D.*



In

In the very same manner, a circle may be described to touch three given circles; the Problem amounting to no more than, *To find a point from whence lines, drawn to three given points, shall have given differences*: Since a point so found, will always be the center of the required circle, as well when the three given circles are to touch that circle inwardly, as when they are all required to touch it outwardly.

The End of the GEOMETRICAL CONSTRUCTIONS.

NOTES

NOTES

GEOMETRICAL and CRITICAL

ON THE

Elementary Part of this WORK.

AXIOM 10. *Book I.* What is here laid down, as an Axiom, would, more properly, have been made a proposition, had it admitted of such a demonstration, as is perfectly consistent with geometrical strictness and purity. But the laying of one figure upon another, whatever evidence it may afford, is a mechanical consideration, and depends on no *Postulate*.

Theor. 4. and 5. Book I. There is scarce any thing more obvious to sense, and at the same time more difficult to demonstrate, than the first, and most simple properties of parallel lines. Even when we have (in *Theor. 4.*) proved the possibility of the existence of such lines, we cannot from thence infer, that their distance from one another is every where the same; without having recourse to an Axiom, which, though very evident to sense, cannot be demonstrated. These difficulties wholly arise from our not having any properties, previously demonstrated, whereby the progress of a right-line, produced out, can be traced, with respect to its distance from some other right-line assigned;

assigned; nothing less being required here, than the proportionality of the sides of equiangular triangles; which is not proved before the middle of the Fourth Book, and which depends upon these very principles.

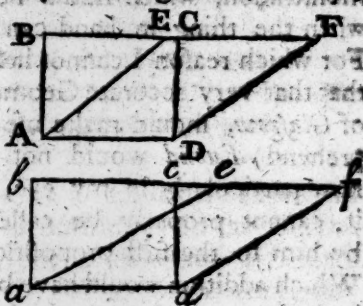
Schol. to Theor. 5. Book I. As there are several conditions requisite to make up the definitions of a rectangle and square, it was necessary to shew here, that the several properties ascribed to those figures, are not incompatible one with another. *Euclid* is very strict in this particular, and never undertakes to demonstrate any thing relating to a figure, till he has proved the possibility of the existence of such figure by an actual construction.

Theor. 22. Book I. This proposition, which is not in *Euclid*, is of considerable use, being often wanted in determining the *Maxima* and *Minima*, in mathematical enquires.

Theor. 27. Book I. This Theorem, though not in *Euclid*, is also very useful, at least, to our design: by it we are not only enabled to divide a right-line into any number of equal parts, without the help of proportions, but also to demonstrate that most important proposition, *That the homologous sides of equiangular triangles are proportional.* It is true, the method pursued here, is not exactly conformable to the idea of proportions delivered in the 6th Def. of *Euclid's* 5th Book. But, even in that light, the demonstration will be equally easy, without inferring it from the proportionality of triangular spaces.

Theor.

Theor. 1. Book II. This proposition, which is not mentioned by *Euclid*, may be thought unnecessary, but it must either be demonstrated, or assumed, as it is wanted in almost every proposition of the second book: By means of it, we also arrive at a general, and very easy demonstration of that important Theorem,



That all parallelograms (AEFD, aefd) which stand upon equal bases, and have equal altitudes, are themselves equal. For, when it is known that these parallelograms AEFD, aefd, are equal, respectively, to rectangles ABCD, abcd, of the same base and altitude (which is proved in Prop. 2.) it is also manifest that they must be equal to one another, as their equal rectangles ABCD, abcd are shewn (by *Theor. 1.*) to be equal, the one to the other. This determination is more general than that given by *Euclid*, in the 36th Prop. of his first book; where he demonstrates the equality of parallelograms, whose equal bases are in the same straight line: Which may, perhaps, be thought sufficient for the whole; because, if the bases are not in the same right-line, one of the two figures may be conceived to be removed, and so placed, that its base shall be in the same right-line with the base of the other.—But, that these were not *Euclid's* sentiments, is evident from this; he hints at no such thing: And had he approved of this sort of demonstration, his 36th Prop. would have been entirely useless; as nothing more (after the 35th) would be necessary, in order to a general demonstration, than barely to place one base upon the
S other.

other. But it is certain, that this is a kind of demonstration, which *Euclid* never has recourse to, when the thing in hand can be done without it. For which reason I cannot help wondering a little, that that very accurate Geometer, *Professor Simpson of Glasgow*, should make use of it, where (I apprehend) *Euclid* would not. The place, I have now particularly in my eye, is the addition (for it cannot properly be called a corollary) made by him to the first proposition of the sixth book: Which addition would have been quite unnecessary, had what is above remarked, respecting the equality of parallelograms, been fully established in the second book: For then the demonstration, that parallelograms, having equal altitudes, are as their bases, would nothing differ from that whereby parallelograms, standing between the same parallels, are proved to be in proportion as their bases.

Theor. 9, 11, 12, 13. Book II. These four Theorems, tho' not in *Euclid*, are of very considerable use, particularly the two first of them.

Theor. 1. Book III. This easy proposition is added in order to give the learner a proper, and more precise idea of the quantity of an angle, and of its division in practical uses.

Theor. 2, 3, 4, 5, 6, 7, 8. Book III. These seven propositions comprehend all that is most material in the first seventeen Theorems of *Euclid's* 3d book.—As there is no where, in this author, so long a run of propositions together that are less entertaining to learners, or of less real importance, than the greater part of the said Theorems; I thought it would be of use to reduce the substance of them into a less compass: And I flatter myself, that I have not succeeded ill, in this particular.

Theor.

Theor. 24, 25, 26, 27, 28. Book III. These Theorems, which are all of very considerable use, will not, I flatter myself, appear less plain by being proved independent of proportions, as the demonstrations here given are, not only more concise, but depend also upon fewer principles.

Def. 12. and 13. Book IV. The explication here given, is not strictly conformable to the idea of ancient *Commentators*, but is delivered in a sense somewhat more general. With them, *the composition and division of ratios*, extends to those cases only, where the sum, or difference of each antecedent and its consequent, is compared with the consequent. When the antecedent is compared with its excess above the consequent, this they call the *conversion of ratios*. But in such cases where the antecedent is less than the consequent, and where the sum, or difference of the antecedent and consequent is required to be compared with the antecedent; it does not appear that any terms have been given to signify such a comparison. *Professor Simpson* thinks, that the definitions we have, are not *Euclid's*, but an addition by *Theon*; which to me appears highly probable: This at least seems clear, that these definitions ought, either to have been extended to a greater number, or else to have been rendered more general. For this reason I have, after the example of modern Geometers, given the signification of those terms, so as to include all the several cases: And this, I thought, might be done with the greater propriety, as the truth of whatever is here understood, depends upon the same demonstration.

Theor. 20, 21, 22, 23. Book IV. All these Theorems, tho' not in *Euclid*, are of considerable use:

by the two former, the demonstration of several others is rendered more easy; and the latter have been applied, to good purpose, in the analytical determination of some difficult geometrical problems.

Prob. 3, 4, 5, 6. Book V. In the demonstrations here given, it may be thought, that I have affected an unnecessary exactness, by making them depend on *Axioms* alone. But I was willing that these fundamental propositions should have the same foundation and evidence, as they appear to stand upon in *Euclid*; without referring to any thing derived in virtue of the 4th Postulate. But, whether I judged well, or ill, in this particular, is of little consequence, as the demonstrations here given, are not less plain, and but very little longer, than they otherwise would have been.—The Problems themselves might, indeed, have been given along with the Theorems, as they became necessary, according to the method pursued by *Euclid*; whereby any objection, of this sort, might have been avoided. But, besides some small convenience to the learner, there is a real advantage in having the Problems altogether, after the Theorems; since, from a great choice of properties, ready demonstrated to our hands, we are often able to arrive at a shorter and better construction, than could possibly be given from such properties alone as are antecedent to *Euclid's* solution of the same Problem; his method of writing having obliged him to introduce the leading Problems as soon as possible, in order to evince and establish the consistence of his definitions, and to open his way in a regular manner to the many useful Theorems thereon depending. And it is for this reason alone, that many of his Constructions are not so well adapted to practice, as those in common use.

Upon which account, some have been precipitate enough to blame him; not seeing, or considering, that such constructions, tho' not suited to answer every purpose, were the most proper for his plan, and the best that could be given in the places where they stand.

Prob. 8. Book V. The reasoning in this proposition, to prove that the two circles will cut each other, may, to some, appear needless. *Professor Simpson* (at p. 359. of his *Euclid*, 4th Edit. 1756) has been a little severe upon me, on this head, for attempting to supply, what I thought, a small defect in *Euclid*. "Who is so dull (says he) tho' only beginning to learn the Elements, as not to perceive that the circle described from the center F, &c." It is not without a real concern that I here see this able Geometer drop his own character so far, as to express himself in a manner so very *ungeometrical*. If the thing is, indeed, easy to be perceived, it must be so, either, as an immediate object of the senses, that is, in plain terms, by inspection; or else it must be in consequence of geometrical reasonings antecedent to the thing itself. Now I am clear that he would not be thought to mean the former; and as to the latter, nothing had been given from whence the evidence of the inference could be so clearly seen: For, tho' it is proved, that any two sides of every triangle are greater than the third side, it would be absurd to urge that conclusion in the case before us; because the question here, is, whether a triangle, under certain specified conditions, can, or cannot be formed? and, therefore, to conclude any thing from the properties of triangles, would be ridiculous, and nothing less than begging the question. That the determination proposed limits the Problem, nobody will dispute: But then, is it not necessary that

this should be proved, in Elements of Geometry, where a reason for every thing is, or ought to be given? From this consideration, I cannot entirely approve of the emendation proposed by this Editor to *Euclid's* 24th Prop. Book I. For, tho' the addition there made, does indeed restrain the proposition to one case; yet this ought to have been demonstrated, by shewing that the point F (vid. p. 29.) must, in consequence of such restriction, necessarily fall below the line EG; but this is not done. Many other instances might be produced to shew, that this gentleman, who often appears a little too hasty and severe in his censures, is not, himself, every where equally guarded. In Prop. I. Book III. he bids you to draw a straight line within a circle, without specifying that it must terminate in the circumference; and, what is a great deal worse, he here very improperly uses the word *within*; when the proposition itself is laid down in order to prove, in the subsequent one, that such line must necessarily fall *within* the circle. These are, it is true, but little matters; but less than these have fallen under this gentleman's notice. At p. 358, Mr. *Clairaut* is glanced at, for an inadvertency of this sort. And, in the note at the head of p. 415, it is said, "The words, *for a straight-line cannot meet a straight-line in more than one point*, are left out, as an addition by some unskilful hand; for this is to be demonstrated, not assumed." Now can it possibly shew any want of skill in an editor, to refer to an axiom which *Euclid* himself had laid down (Book I. N°. 14.) and not to have demonstrated, what no man can demonstrate?

Prob. 16, 17, 18. *Book V.* These three Problems, tho' not so frequently wanted as some of the preceding ones, are nevertheless of very considerable use. The two last of them are the same,
in

in effect, with the 28th and 29th of *Euclid's* 6th book; but are here put down in a manner rather more commodious.

Axiom, p. 131. This Axiom is substituted instead of the common definition of equal solids; which, I really think, is too bad to be the work of *Euclid*. "It is not a definition, but a proposition, whose truth or falshood ought (as a very judicious writer observes) to be demonstrated, not assumed." Neither is it at all conformable to *Euclid's* manner of writing, where he establishes the foundation whereon the equality of plane figures is grounded; which he does, not by means of a definition, but from the application of two of the most simple figures to each other; so that, from the coincidence of their bounds, their equality may appear manifest. And this method we have pursued in treating of solids; without which a clear and distinct idea of their equality can be no more obtained, than of the equality of plane figures independent of the 4th Prop. of the first book, which is our 10th Axiom. I should have said a good deal more on this head; but I find that *Professor Simpson* has already placed this matter in so strong and clear a light, as to render any farther apology, or comment, unnecessary here. Tho' I must confess, that, had this gentleman's work come into my hands * before the elementary part of my own had

* This did not happen till the middle of November 1759; when being in town, in company with my bookseller, and being pressed by him to finish; I acquainted him that every thing was actually ready except the Preface, which would cost me some pains, since it would be necessary to obviate some objections, particularly with regard to the reasons for my rejecting *Euclid's* definition of equal solids, and building upon a different foundation. On which, he immediately let me know, that Mr. Rob. Simpson had already cleared

had been intirely printed off, the definition of similar solids, which I have given, from *Euclid*, would have been delivered under a form somewhat different. For, tho' it involves no absurdity, as it now stands, yet there are certain cases (but such indeed as do not occur in any Elements of Geometry) where it will not afford the precise idea it ought to convey.

Postulatum, p. 131. The unsatisfactory and inconclusive demonstration given to *Euclid's* 2d Prop. Book XI. (by *Euclid* himself, or some less skilful Editor) seemed to render something of this sort necessary. In that demonstration it is taken for granted, that one part of the triangle, at least, must be in the same plane. But it has been very justly observed, that a curve surface may be bounded by three right-lines: Nor does it seem easy to form a clear idea, that even a part of any one of the three lines will be in the same plane with one of the others, unless by conceiving a plane to be turned about upon the one, till it meets with, or falls upon some point in the other. And I have the satisfaction in this particular, to see my sentiments exactly agree with those of a very good judge, whose name, I have, more than once, had occasion to mention in these notes. It is true, he makes that a Theorem, which I lay down as a Postulate. But, since a plane can no more be turned about upon a line, than a line can be drawn from one point to another, it seemed to me, that the one

up that point; and expressed his surprise that a work of so much repute, wherein (he told me) my own name was more than once mentioned, had not come into my hands. A copy of which I received from him the next morning. In consequence whereof I changed my first design of writing a long Preface; thinking it would be better to give what I had to offer, in notes, after the example of this Editor.

was as properly a Postulate, as the other. However, whether this be, or be not allowed, is immaterial, as the degree of evidence is precisely the same. There is, indeed, one reason, why this Theorem, or Postulate, ought to have preceded that gentleman's demonstration of Prop. 1. Book XI; It is there wanted: For, in the Corollary, on which that demonstration is made to depend, the lines AB, BD, BC, are supposed to be all in the same plane; which ought by no means to be assumed in the first of the 11th. Euclid's 10th Axiom, which that Corollary is intended to supply, and by which the proposition is usually demonstrated, is not limited by any such restriction.

Theor. 12. Book VII. This proposition is added on account of its use, being the foundation on which the whole *art of perspective* in a manner depends.

Theor. 25. Book VII. From this Theorem, which is very extensive in its application, several others of considerable note may be deduced: one, or two of which, for the sake of the learner, I shall here derive, and put down by way of example.

Let A, B, C, D, denote four lines in continual proportion.

Then, $\left\{ \begin{array}{l} A : B :: C : D, \\ A : B :: B : C, \\ A : B :: A : B, \end{array} \right\}$ it follows (from *Theor. 25.*) that

$A^3 : B^3 :: CBA : CBD :: A : D$ (by 22. 7.) or that, of four lines in continual proportion, the cube of the first is to the cube of the second, as the first line is to the fourth.

Again, let $A : a :: B : b :: C : c$ (where A, a; B, b; C, c, may be supposed to represent the homologous sides of two similar parallelepipeds.)

Then,

Then, $\left\{ \begin{array}{l} A : a :: B : b \\ A : a :: C : c \\ A : a :: A : a \end{array} \right\}$ it also follows,

that $A^3 : a^3 :: BCA : bca$; or, that similar parallelepipeds are to one another, as the cubes of their homologous sides.—The proportionality of similar parallelepipeds, described upon proportional lines, is also included in the same Theorem; being no other than that case of it, where the proposed ratios are all equal.

Theor. 5. Book VIII. The demonstration of this Theorem might have been delivered under a form somewhat different, by assuming two other solids (without regard to figure) the one less, and the other greater than the proposed parallelepipedon IP, and proving that the cylinder must, *also*, be greater than the one, and less than the other: which is done by means of *Lem. 1.* that is, by taking Pm, or Pt, so small a part of IP, as to be less than the difference between the given parallelepipedon and either of the said solids: from whence the demonstration will proceed on, in the same manner in which we have given it. But, as these additional considerations would have increased the number of schemes, and lengthened the process, without adding one jot to the degree of evidence, it was thought proper, for the sake of the learner, to omit them.

Theor. 8. Book VIII. This Theorem is not so useful as the Corollaries that follow from it, which are all of very great importance: In the 3d and 4th of them, the proportion of all kinds of prisms and pyramids is assigned, without the assistance of the usual demonstrations given for this purpose; which, tho' sufficiently evident in themselves, are often found a little perplexing to learners, on account

count of the schemes, wherein so great a number of lines is necessary.

Nothing, in the course of these notes, has been said relative to the Theorems on proportions, tho' so nice and critical a subject, and tho' the method I have therein pursued may stand in need of some apology. But, indeed, the whole of what I have to offer on this head, was too much to be comprized in the compass of one single note, and could not so properly be delivered in several detached ones: For which reason, I shall here throw together all that I have to advance on that subject.

There are two objections that may be brought against the method in which proportions are treated of in this work; the one, grounded on the impossibility of dividing every magnitude into equal parts; and the other, on the incommensurability of two or more magnitudes of the same kind, when compared with each other. The first of these objections appears, to me, to have very little weight. For, tho' a magnitude may be so constituted, that the division of it into an assigned number of equal parts cannot be, *actually*, effected by any geometrical construction; yet it is no less evident, for that reason, that every such magnitude has really its third, fourth, or other assigned part, tho' we are at a loss how to take it; or, in other words, it seems very clear to conceive, that in every proposed magnitude, whatever its figure may be, a less magnitude is contained, which, repeated an assigned number of times, shall be equal to the magnitude given. If, as the most rigid judges allow, every plane figure is equal to *some* square, and every solid equal to *some* parallelepipedon; then the parts of the square, or parallelepipedon, which are actually determinable by a geometrical

metrical construction, will also be like parts of the figure first proposed, and such as we conceive to be taken.

The other objection, depending on the incommensurability of magnitudes, is a matter of real difficulty; which we have taken some pains to obviate, in the *Scholias* to our 3d and 7th Theorems. *Euclid*, himself, seems to have been not a little embarrassed with it, if we may be allowed to judge from the different methods he has left us in his 5th and 7th books; the former whereof, which is suited to include the business of incommensurables, being nothing near so easy and natural as the latter: It has, it is true, the advantage of being general; but, that the principles whereon it is grounded, are neither so simple, nor so evident as might be wished for, the many disputes about them, since *Euclid's* time, by Geometers of the first rank, will in a great measure evince. And farther, it seems sufficiently plain, from *Euclid's* own authority, that he himself was not intirely pleased with his own performance on this head; or that he was convinced (at least) that it had not every advantage: For, otherwise, it will be very difficult to account for his having demonstrated many things in his 7th book, by another method, whose demonstrations had been actually given before, in the 5th, under a different form. For these reasons, when I see the extravagant commendations that have been lavished on this 5th book of *Euclid*, I am no farther convinced by them, than that great men may sometimes launch out too far in behalf of opinions which they have adopted. And I believe that, whoever has read the notes on the 5th book, by that great restorer of *Euclid*, *Professor Simpson*, will be apt to conclude, that those high encomiums are a little misapplied. Indeed, if all that is advanced in those notes be allowed of, I think the author of them has proved

too much; and this superb fabric of proportions, reared with so much art, stands upon a tottering foundation. It is not by choice that I go out of my way to play the critic; but as the writers against the vulgar and indistinct notion of proportions (as they term it) are very severe in their censures, and assume a great superiority, from the boasted accuracy of their reasonings, it may be necessary to shew my reader, that, tho' what he is here taught on proportions, is liable to some objections, the method which some so greatly prefer, has also its difficulties; and that there are other objections to it besides its obscurity. And this I shall make appear from this learned Commentator's own authority and concessions; and in order thereto, shall first refer to his note on Prop. 10. which proceeds thus: "It was necessary to give another demonstration of this proposition, because that which is in the Greek and Latin, or in other editions, is not legitimate. For the words greater, the same, or equal, lesser, have a quite different meaning when applied to magnitudes and ratios, as is plain from the 5th and 7th definitions of Book 5. by help of these let us examine the demonstration of the 10th Prop. which proceeds thus, &c." He then goes on, in a long note, to shew the insufficiency of a demonstration, which had been received, by all, as perfectly genuine and satisfactory; and at last comes to this conclusion. "Wherefore the 10th Proposition is not sufficiently demonstrated. And it seems, that he who has given the demonstration of the 10th Proposition, as we now have it, instead of that which Euclid or Eudoxus had given, has been deceived in applying what is manifest, when understood of magnitudes, unto ratios, viz. that a magnitude cannot be both greater and less than another. That those things which are equal to the same are equal to one another,

" is

“ is a most evident Axiom when understood of
 “ magnitudes, yet *Euclid* does not make use of it
 “ to infer that these ratios which are the same to
 “ the same ratio, are the same to one another ;
 “ but explicitly demonstrates this in Prop. 11. of
 “ Book 5. The demonstration we have given of the
 “ 10th Prop. is no doubt the same with that of
 “ *Eudoxus* or *Euclid*, as it is immediately and di-
 “ rectly derived from the definition of a greater
 “ ratio, viz. 7th of 5.”

Here the weight of the objection rests on its not having been proved, that, of three given magnitudes A, B, C, the ratio of A to C could not, at the same time, be both greater and less than that of B to C. But, if in the demonstration, here rejected as insufficient, there is any real flaw, it is chargeable on the definition of a greater and less ratio, as the reasoning from it, is clear, strong, and perfectly scientific. And I would seriously ask the Contemners of *the vulgar and confused notion* of proportions, if a definition, by which it cannot be known, whether the ratio of the first to the second of four given magnitudes, may not, at the same time, be both greater and less, than that of the third to the fourth, is really calculated to afford those very accurate ideas they pretended to? This Commentator has too much penetration not to be aware of the force of this objection, which he has attempted to obviate in one particular case. But the new proposition given by him, for that purpose, ought to have preceded the 10th, and to have been demonstrated, independent of it. This he also seems apprized of, when he says, that “ it cannot be easily demonstrated without the 10th, as he that “ tries to do it will find.” But, be that as it will, I am not at all clear that his “ demonstration of “ the 10th, is the same with that of *Eudoxus* or “ *Euclid*.” *Euclid*, or (if you please) *Eudoxus*, does
 never

never (that I know of) refer to any definition, till it has been proved, either by an actual construction, or by some demonstration previous to that in hand, that such definition involves no absurdity, or conditions that are incompatible one with another. If, therefore, it was conceived possible, that the definition of a greater and less ratio, could involve so great an absurdity, as that, *by it*, the ratio of A to C might at the same time be both greater and less than that of B to C; this point, according to the method prescribed by *Euclid*, ought to have been cleared up, not by means of propositions derived in virtue of that very definition, but by others antecedent thereto, and independent thereupon. And, to me, the 8th Prop. seems the proper place for the doing of this, where it might be easily introduced, either in the Prop. itself, or by way of Corollary. It is there proved, that if, of three magnitudes A, B, C, the first A is greater than the second B, then certain equimultiples of A and B may be taken such, that being compared with some multiple of C, the multiple of A shall be greater, and that of B less than the said multiple of C. Whence, by the definition of a greater ratio, the ratio of A to C is greater than that of B to C. To which might be added—And because A is greater than B, any multiple whatsoever of A must be greater than the same multiple of B; and, consequently, no equimultiples whatsoever, of A and B, can possibly be so taken, that the multiple of A shall be equal to, or less than some multiple of C, and that of B greater than the multiple of C: for, if the multiple of B be greater than the multiple of C, the multiple of A, which is greater than that of B, must also be greater than the multiple of C. Wherefore the ratio of A to C cannot (by the definition) be less (as well as greater) than the ratio of B to C.

But, notwithstanding all that has been proved on this head, either here, or by that gentleman himself,

self, the same objection occurs again in Prop. 13. where it remains in its full force. For, tho' it be allowed, that "there are some equimultiples of C and E, and some of D and F, such, that the multiple of C is greater than the multiple of D, but the multiple of E not greater than the multiple of F;" yet it is not demonstrated, nor in any sort shewn, that other equimultiples of those quantities cannot be taken such, that the very contrary shall happen.—If the demonstration of the 10th Prop. has been justly rejected by this gentleman himself, as insufficient, because the impossibility of a contrary conclusion had not been shewn, can it be thought that this 13th Proposition is, at this day, sufficiently demonstrated, where the same objection occurs, and that in a much greater latitude? I have a much better opinion of this Editor's discernment, than to imagine, that his passing this matter over in silence, proceeded from his not being aware of the difficulty; but it seems to me, that his great dislike to the *vulgar* idea of proportion (so often testified in the course of his notes) would not permit him to borrow any thing from thence, however evident, and though this objection, that strikes deep at the very root of proportions, might by means thereof be very easily removed. I say, the very root of proportions is deeply struck at in this objection; because both the alternation and equality of ratios (*ex equali sc. dist.*) are grounded on the said 13th Prop. and which, therefore, till the objection is removed, must be allowed to stand upon an uncertain foundation.

The principle hinted at above, whereby the difficulty might be obviated, is, that if a magnitude of any kind be given, or propounded, there may (or can) be another magnitude of the same kind which shall have to it any ratio assigned. This assumption Mr. *Professor* will by no means admit of
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(tho' *Euclid* himself, in Prop. 2. of his 12th book, has used it); and, in a long note on Prop. 18. is angry with *Clavius* for having recourse to it: affirming, "that the demonstration (given by means thereof) is of no force;" and that "the thing itself cannot (as far as he can discern) be demonstrated by the preceding propositions, so far is it from deserving to be reckoned an Axiom, as *Clavius*, after other Commentators, would have it." That the assumption cannot be generally demonstrated by the preceding propositions (nor even by all the propositions in the *Elements*) I readily assent to: but then, because a thing, exceedingly obvious in its own nature, cannot be demonstrated; is it therefore less proper for an Axiom? I should rather take the other side of the question, and maintain that nothing ought to be made an Axiom, which can be demonstrated. But we are not, it seems, allowed to have any idea of proportion but what is contained in the sixth and 8th (or, as this Author makes them, the 5th and 7th) definitions of *Euclid's* 5th Book. And, in his note on the new Prop. marked A, He is again displeased with *Clavius*, for thinking it sufficiently evident, from the nature of proportionals, that if, of four proportional magnitudes, the first antecedent is greater than its consequent, the second antecedent will also be greater than its consequent. "As if there was" (says he) any nature of proportionals antecedent to that which is to be derived and understood from the definition of them." Now, I cannot help thinking, with *Clavius*, that there was a nature, or idea of proportion antecedent to that given in the 6th and 8th definitions of *Euclid's* 5th book: For, that mankind, long before the time of *Euclid*, had some way to shew, or express, in what degree one magnitude was greater or less than another, cannot be doubted: And this was

the first, and natural idea of proportion: And I look upon those definitions, as refinements, only, on the simple and natural idea, in order to take in the business of incommensurables; whereby the original notion is so much obscured, that it requires some skill, even to see that it is at all contained in these definitions. I intirely agree with this gentleman, that every demonstration ought to be strictly derived from principles before established: But then, whether it is more eligible, to have recourse to an Axiom founded (as all other Axioms are) on the evidence of sense and reason, or to an obscure and perplexed definition, which may, for any thing that has been proved to the contrary, involve an absurdity?

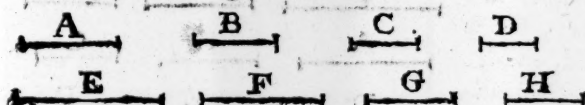
That there is something very ingenious and subtle in the doctrine of proportions, as delivered in *Euclid's* 5th book, cannot be denied. All that I contend for, is, that the principles on which it is built are *obscure*, and not so firmly established, as to authorize its partizans to assume that great superiority they lay claim to, in point of geometrical strictness.

I have intimated above, that the principle is rejected, by which the consistence of the definition of a greater and less ratio might be established, without much difficulty: But I would not be thought to mean, that the same thing cannot possibly be effected any other way, because I am satisfied that it may be done from the consideration of multiples alone: But a demonstration of this sort is not easy.—Were I to treat of proportions from the plan laid down in the 5th book of *Euclid*, I would entirely reject the 10th and 13th propositions, and every thing else founded on the definition of a greater and less ratio, as being of no other use in the Elements, than to open the way to those important Theorems on the alternation and equality of ratios; which may be better demonstrated without them, from the definition of equal ratios

tios alone; which, from the conditions of it, can admit of no absurdity, and whose consistence is evinced in Prop. 15. and still more clearly in the first of the sixth.

In the 14th of the 5th, whereon the alternation of ratios is grounded, it is necessary to demonstrate, "That if, of four proportional magnitudes, of the same kind, A, B, C, D, the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, lesser." Which may be very easily done, independent both of the 10th and 13th, in the manner following.

First, let A be greater than C.



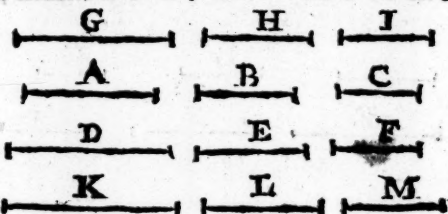
Of A and C (*by Prop. 8.*) let such equimultiples be taken, that the multiple of A shall be greater, and that of C less, than some multiple of B; let E and G be any two such equimultiples of A and C, and F the multiple of B; so that E shall be greater than F, and G less than F; and let H be the same multiple of D, as F is of B. Therefore, because E and G are equimultiples of the first and third, and F and H also equimultiples of the second and fourth; and seeing that, (*by Hyp.*) E is greater than F; it is evident, from the definition of equal ratios, that G must likewise be greater than H: Therefore much more shall F (which exceeds G) be greater than H; whence also B shall be greater than D (*by Ax. 4.*) B and D being like parts of F and H.

When A is less than C, it will be demonstrated in the same manner, that B is also less than D. But when A is equal to C, no new demonstration is necessary; since neither the 10th nor the 13th have any thing to do in this case.

Again, in the 20th Prop. (in which the 10th and 13th also enter) we are to prove, "That if there be
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" three magnitudes (A, B, C) and other three (D, E, F) which taken two and two have the same ratio (A : B :: D : E, B : C :: E : F) if the first (A) be greater than the third (C), the fourth (D) shall be greater than the sixth (F); and if equal, equal; and if less, lesser." Which may likewise be done, without the assistance of either the 10th or the 13th, in the same manner, above specified.

For, if A be greater than C; then of A and C (by Prop. 8.) such equimultiples may be taken, that the multiple of A shall be greater, and that of C less, than some multiple of B; let G and I be two such equimultiples of A and C, and let H be the multiple of B, so that G shall be greater than H, and I less



than H; moreover take L the same multiple of E, as H is of B; and K and M the same equimultiples of D and F, as G and I are of A and C. Therefore, since of the four proportionals A, B, D, E, equimultiples G, K of the first and third, and equimultiples H, L of the second and fourth, are here taken, it is manifest, from the definition of equal ratios, seeing G is greater than H (by Hyp.) that K must also be greater than L. And in the very same manner, because B, C, E, F, are proportionals, and H is greater than I, L will likewise be greater than M: Therefore much more shall K, which exceeds L, be greater than M. And consequently (by Ax. 4.) D shall also be greater than F. — When A is less than C, the demonstration is the same: The other case, when A is equal to C, does not require, nor indeed admit of any improvement.

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